Design and Analysis of a Gain-scaling Controller Applied to a Chain of Integrator System with Measurement Noise on Feedback Sensor

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Abstract – A control problem of a chain of integrator system with measurement noise on feedback sensor is considered. We propose a gain-scaling controller for compensating measurement noise of feedback sensor. Because control systems operate via feedback sensor's signal, the measurement noise in sensor' signal results in performance degradation or even system failure. Therefore, control systems often demand on compensating measurement noise. Our controller is equipped with a compensator and gain-scaling factor in order to reduce the effect of measurement noise on feedback sensor.

Key words – *measurement noise*; *gain-scaling factor*; *chain of integrator system*

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1 Introduction

Control systems operate via measurement data. It is usually assumed that the measured signals are clean, i.e., there is no noise in feedback signals. If measurement noise exists in sensor's signal, it has influence on controller output and then it may even generate whole system's failure. Therefore, we need to compensate measurement noise in controller design[1-3]. In Ref. [1], the problem of feedback sensor has been formulated and restricted measurement noise has been compensated. In Ref. [3], measurement noise has been compensated by using the Proportional-Integral-Derivative(PID) controllers based on optimization without analysis.

In this paper, we suppose three types of measurement noise which are DC, AC, and DC and AC component. These noises are added in controlling a chain of integrator system. The measurement noise is compensated by using gain-scaling controller with low pass filter.

2 Formulation

We consider a chain of integrator system given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &= u \\ y &= x_1, \end{aligned} \tag{1}$$

where $\boldsymbol{x}=[x_1,\cdots,x_n]^{\mathrm{T}}\in R^n$ is the state, $\boldsymbol{u}\in R$ is the input and $\boldsymbol{y}\in R$ is the output. Ideally, a state feedback controller can be implemented by

$$\boldsymbol{u} = \boldsymbol{Kx}, \tag{2}$$

where $\boldsymbol{K}=[k_1,\cdots,k_n]$. However, controller (2) is only available in the ideal environment in the absence of measurement noise. If the feedback information includes measurement noise, the controller (2) becomes

$$U = K\chi, \tag{3}$$

Where $\chi = x + \boldsymbol{s}(t)\in R^n$ is measurement state and $\boldsymbol{s}(t)=[s_1(t),\cdots,s_n(t)]^{\mathrm{T}}\in R^n$ is measurement noise. So bounded measurement noise produces bounded states[4-5] and system regulation becomes difficult. Therefore, we propose the following gain-scaling control law[6] equipped with compensator[7] for bounded measurement state.

$$u = K(\varepsilon)\chi * \mathrm{e}^{(k_{n+1}t)/\varepsilon}, \tag{4}$$

where $K(\varepsilon) = [k_1/\varepsilon^{n+1},\cdots,k_n/\varepsilon^2], \varepsilon > 0$, $*$ denotes convolution operation. Here, we let $u = x_{n+1}$, then the chain of integrator system (1) is transformed into

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n+1} &= v \\ y &= x_1 \end{aligned} \tag{5}$$

and

$$v = \bar{K}(\varepsilon)\bar{\chi}, \tag{6}$$

where $\bar{K}(\varepsilon) = [k_1/\varepsilon^{n+1},\cdots,k_{n+1}/\varepsilon]$ and $\bar{\chi} = [\chi_1,\cdots\chi_n, \chi_{n+1}] \in R^{n+1}$.

3 Analysis

We apply Laplace transform to Eq. (5) in order to

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analyze the response of Eq.(5). Then, it becomes

$$\begin{aligned} sX_1(s) - x_1(0) &= X_2(s) \\ sX_1(s) - x_1(0) &= x_3(s) \\ &\vdots \\ sX_n(s) - x_n(0) &= X_{n+1}(s) \\ sX_{n+1}(s) - x_{n+1}(0) &= \frac{k_1}{\varepsilon^{n+1}}(X_1(s) + S_1(s) + \\ &\frac{k_2}{\varepsilon^n}(X_2(s) + S_2(s)) + \cdots + \\ &\frac{k_n}{\varepsilon^2}(X_n(s) + S_n(s)) + \\ &\frac{k_{n+1}}{\varepsilon}X_{n+1}(s) \\ Y(s) &= X_1(s). \end{aligned} \tag{7}$$

Thus, system's output is

$$\begin{aligned} Y(s) = &\frac{\delta_{n+1,1}(s)}{\delta_{n+1}(s)}x_1(0) + \frac{\delta_{n+1,2}(s)}{\delta_{n+1}(s)}x_2(0) + \cdots + \\ &\frac{\delta_{n+1,n+1}(s)}{\delta_{n+1}(s)}x_{n+1}(0) + \frac{k_1}{\delta_{n+1}(s)\varepsilon^{n+1}}S_1(s) + \\ &\frac{k_2}{\delta_{n+1}(s)\varepsilon^n}S_2(s) + \cdots + \frac{k_n}{\delta_{n+1}(s)\varepsilon^2}S_n(s), \end{aligned} \tag{8}$$

where $\delta_{n+1}(s), \delta_{n+1,1}(s), \cdots, \delta_{n+1,n+1}(s)$ are given by

$$\begin{aligned} \delta_{n+1}(s) &= s^{n+1} - \frac{k_{n+1}}{\varepsilon}s^n - \frac{k_n}{\varepsilon^2}s^{n-1} - \cdots - \frac{k_2}{\varepsilon^n}s - \frac{k_1}{\varepsilon^{n+1}} \\ \delta_{n+1,1}(s) &= \frac{(\delta_{n+1}(s) + \frac{k_1}{\varepsilon^{n+1}})}{s}, \\ \delta_{n+1,2}(s) &= \frac{(\delta_{n+1}(s) + \frac{k_2}{\varepsilon^n})}{s}, \\ &\vdots \\ \delta_{n+1,n+1}(s) &= \frac{(\delta_{n+1}(s) + \frac{k_{n+1}}{\varepsilon})}{s} = 1. \end{aligned} \tag{9}$$

If $\delta_{n+1}(s)$ becomes Hurwitz polynomial, we can use final value theorem on Eq.(8). Then the factor of $S_n(s)$ which has unknown pole remains[8] as

$$\begin{aligned} &\frac{k_1}{\delta_{n+1}(s)\varepsilon^{n+1}}S_1(s) + \frac{k_2}{\delta_{n+1}(s)\varepsilon^n}S_2(s) + \cdots + \\ &\frac{k_n}{\delta_{n+1}(s)\varepsilon^2}S_n(s). \end{aligned} \tag{10}$$

3.1 DC measurement noise

We suppose that the measurement noise is represented by DC component.

$$\begin{aligned} s_1(t) &= \alpha_1 & S_1(s) &= \alpha_1/s \\ s_2(t) &= \alpha_2 \Rightarrow & S_2(s) &= \alpha_2/s \\ &\vdots & &\vdots \\ s_n(t) &= \alpha_n & S_n(s) &= \alpha_n/s \end{aligned} \tag{11}$$

After applying Eq.(11) to Eq.(10), using the final value theorem the remaining terms is

$$-\alpha_1 - \frac{\alpha_2 k_2 \varepsilon}{k_1} - \frac{\alpha_3 k_3 \varepsilon^2}{k_1} - \cdots - \frac{\alpha_n k_n \varepsilon^{n-1}}{k_1}. \tag{12}$$

Therefore, if $s_1(t), \cdots, s_n(t)$ are DC component, factor of $S_1(s)$ converges to $-\alpha_1$ regardless of size of ε and the effects of $S_2(s), \cdots, S_n(s)$ are reduced by decreasing ε .

3.2 AC measurement noise

We suppose that the measurement noise is represented by AC component (represented by sine function). Let $s_1(t), \cdots, s_n(t)$ become

$$\begin{aligned} & & S_1(s) &= \frac{\rho_1\omega_1}{s^2 + \omega_1^2} \\ s_1(t) &= \rho_1 \sin\omega_1 t & & \\ s_2(t) &= \rho_2 \sin\omega_2 t \Rightarrow & S_2(s) &= \frac{\rho_2\omega_2}{s^2 + \omega_2^2} \\ &\vdots & &\vdots \\ s_n(t) &= \rho_n \sin\omega_n t & & \\ & & S_n(s) &= \frac{\rho_n\omega_n}{s^2 + \omega_n^2} \end{aligned} \tag{13}$$

After applying Eq.(13) to Eq.(10), using the final value theorem the remaining term is

$$\begin{aligned} &\frac{\rho_1 k_1}{\varepsilon^{n+1}}\left(\frac{\beta_{n+1,1}\omega_1 - \gamma_{n+1,1}s}{\beta_{n+1,1}^2 + \gamma_{n+1,1}^2}\right)\left(\frac{1}{s^2 + \omega_1^2}\right) + \\ &\frac{\rho_2 k_2}{\varepsilon^n}\left(\frac{\beta_{n+1,2}\omega_2 - \gamma_{n+1,2}s}{\beta_{n+1,2}^2 + \gamma_{n+1,2}^2}\right)\left(\frac{1}{s^2 + \omega_2^2}\right) + \cdots + \\ &\frac{\rho_n k_n}{\varepsilon^2}\left(\frac{\beta_{n+1,n}\omega_n - \gamma_{n+1,n}s}{\beta_{n+1,n}^2 + \gamma_{n+1,n}^2}\right)\left(\frac{1}{s^2 + \omega_n^2}\right) \end{aligned} \tag{14}$$

where

$\beta_{n+1,n} = \mathrm{Re}(\varepsilon_{n+1}(s)\,|_{s=j\omega_n}, \gamma_{n+1,n} = \mathrm{Im}(\delta_{n+1}(s)\,|_{s=j\omega_n}).$

Therefore, increasing ε and decreasing $k_1, \cdots, k_n$ can reduce the effect of $s_1(t), \cdots s_n(t)$.

3.3 DC and AC measurement noise

If measurement noise contains both DC and AC components, $s_1(t), \cdots, s_n(t)$ become

$$\begin{aligned} s_1(t) &= \alpha_1 + \rho_1 \sin\omega_1 t \\ s_2(t) &= \alpha_2 + \rho_2 \sin\omega_2 t \\ &\vdots \\ s_n(t) &= \alpha_n + \rho_n \sin\omega_n t \end{aligned} \tag{15}$$

After applying Eq.(15) to Eq.(10), using the final value theorem the remaining terms is

$$\begin{aligned} &-\alpha_1 + \frac{\rho_1 k_1}{\varepsilon^{n+1}}\left(\frac{\beta_{n+1,1}\omega_1 - \gamma_{n+1,1}s}{\beta_{n+1,1}^2 + \gamma_{n+1,1}^2}\right)\left(\frac{1}{s^2 + \omega_1^2}\right) - \\ &\frac{\alpha_2 k_2 \varepsilon}{k_1} + \frac{\rho_2 k_2}{\varepsilon^n}\left(\frac{\beta_{n+1,2}\omega_2 - \gamma_{n+1,2}s}{\beta_{n+1,2}^2 + \gamma_{n+1,2}^2}\right)\left(\frac{1}{s^2 + \omega_2^2}\right) + \cdots - \\ &\frac{\alpha_n k_n \varepsilon^{n-1}}{k_1} + \frac{\rho_n k_n}{\varepsilon^2}\left(\frac{\beta_{n+1,n}\omega_n - \gamma_{n+1,n}s}{\beta_{n+1,n}^2 + \gamma_{n+1,n}^2}\right)\left(\frac{1}{s^2 + \omega_n^2}\right) \end{aligned} \tag{16}$$

Here, we can know that the change of size of ε does not reduce the both DC and AC effect at the same time. However, only decreasing $k_2, \cdots, k_n$ can reduce the effect of both DC and AC component of $s_2(t), \cdots, s_n(t)$.

Remark: $\varepsilon, k_1, \cdots, k_n$ should be chosen according to the type of measurement noise. Here, we consider three conditions of measurement noise.

1) (DC type): Factor of $s_1(t)$ converges to $-\alpha_1$ regardless of size of ε, k_1. However, we can reduce effect of $s_2(t), \cdots, s_n(t)$ by decreasing ε and increasing $k_2, \cdots, k_n$.

2) (AC type): We can reduce the effect of $s_1(t), \cdots, s_n(t)$ by increasing ε and decreasing $k_1, \cdots, k_n$.

3) (DC and AC type): We can reduce the effect of $s_1(t), \cdots, s_n(t)$ by decreasing of $k_1, \cdots, k_n$. However, DC component of $s_1(t)$ is not affected.

4 Illustrative example

Consider a system given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_1 &= u \\ y &= x_2 \end{aligned} \tag{17}$$

In Eq. (17), feedback signal's measurement noise is set as $s_1(t), s_2(t)$. Applying Eq. (4), and letting by $x_3 = u$, Eq. (17) is represented by Eq. (5). Here utilizing Eq. (7) and (8), we can obtain the output given by

$$\begin{aligned} Y(s) = &\frac{\delta_{3,1}(s)}{\delta_3(s)}x_1(0) + \frac{\delta_{3,2}(s)}{\delta_3(s)}x_2(0) + \frac{\delta_{3,3}(s)}{\delta_3(s)}x_3(0) + \\ &\frac{k_1}{\delta_3(s)\varepsilon^3}S_1(s) + \frac{k_2}{\delta_3(s)\varepsilon^2}S_2(s), \end{aligned} \tag{18}$$

where $\delta_3(s), \delta_{3,1}(s), \delta_{3,2}(s), \delta_{3,3}(s)$ are given by

$$\begin{aligned} \delta_3(s) &= s^3 - \frac{k_3}{\varepsilon}s^2 - \frac{k_2}{\varepsilon^2}s - \frac{k_1}{\varepsilon^3} \\ \delta_{3,1}(s) &= s^2 - \frac{k_3}{\varepsilon}s - \frac{k_2}{\varepsilon^2} \\ \delta_{3,2}(s) &= s - \frac{k_3}{\varepsilon} \\ \delta_{3,3}(s) &= 1 \end{aligned} \tag{19}$$

Thus, when $\delta_3(s)$ is Hurwitz polynomial, using the final value theorem the remaining term is

$$\frac{k_1}{\delta_3(s)\varepsilon^3}S_1(s) + \frac{k_2}{\delta_3(s)\varepsilon^2}S_2(s). \tag{20}$$

4.1 DC measurement noise

Measurement noise of feedback signal is assumed as DC component in Eq. (20).

$$\begin{matrix} s_1(t) = \alpha_1 \\ s_2(t) = \alpha_2 \end{matrix} \Rightarrow \begin{matrix} S_1(s) = \alpha_1/s \\ S_2(s) = \alpha_2/2 \end{matrix} \tag{21}$$

Thus, since measurement noise has DC component, the remaining term of Eq. (20) becomes

$$\begin{aligned} &\lim_{s\to 0} s\left(\frac{k_1}{\delta_3(s)\varepsilon^3}S_1(s) + \frac{k_2}{\delta_3(s)\varepsilon^2}S_2(s)\right) = \\ &\left.\frac{\alpha_1 k_1}{\delta_3(s)\varepsilon^3}\right|_{s=0} + \left.\frac{\alpha_2 k_2}{\delta_3(s)\varepsilon^2}\right|_{s=0} = -\alpha_1 - \frac{\alpha_2 k_2 \varepsilon}{k_1} \end{aligned} \tag{22}$$

Therefore, if $S_1(s), S_2(s)$ are DC components, the effect of $S_1(s)$ converges to $-\alpha_1$ regardless of size of ε, k_1 and the effect of $S_2(s)$ is reduced if ε is very small value.

This simulation is performed with $k_1 = -6, k_2 = -12, k_3 = -8, \varepsilon = 1$. Fig. 1 illustrates that x_1 is converged as $-\alpha_1$ when $s_1(t) = 1, s_2(t) = 0$.

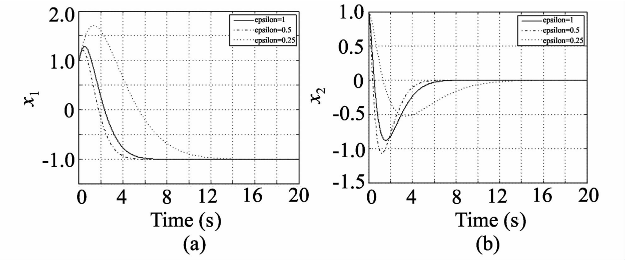

Fig. 1 Illustrates that x_1 is converged as $-\alpha_1$ when $s_1(t) = 1, s_2(t) = 0$: (a) x_1, (b) x_2

Fig. 2 is simulation result when $s_1(t) = 0, s_2(t) = 1$. It shows that x_1 is converged as $-\alpha_2 k_2 \varepsilon / k_1$ and we know that the effect of $s_2(t)$ can be reduced if ε is a very small value.

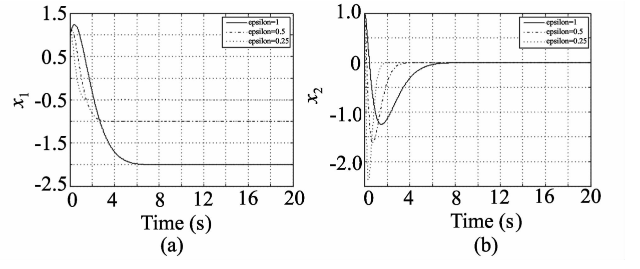

Fig. 2 Simulation result when $s_1(t) = 0, s_2(t) = 1$: (a) x_1, (b) x_2

4.2 AC measurement noise

Measurement noise of feedback signal is assumed as AC component in Eq. (20).

$$\begin{matrix} s_1(t) = \rho_1 \sin\omega_1 \\ s_2(t) = \rho_2 \sin\omega_2 \end{matrix} \Rightarrow \begin{matrix} S_1(s) = \dfrac{\rho_1\omega_1}{s^2+\omega_1^2} \\ S_2(s) = \dfrac{\rho_2\omega_2}{s^2+\omega_2^2} \end{matrix} \tag{23}$$

After applying Eq. (23) to Eq. (20), using the final value theorem the remaining terms is

$$\begin{aligned} &\frac{\rho_1 k_1}{\varepsilon^3}\left(\frac{\beta_{3,1}\omega_1 - \gamma_{3,1}s}{\beta_{3,1}^2 + \gamma_{3,1}^2}\right)\left(\frac{1}{s^2+\omega_1^2}\right) + \\ &\frac{\rho_2 k_2}{\varepsilon^2}\left(\frac{\beta_{3,2}\omega_2 - \gamma_{3,2}s}{\beta_{3,2}^2 + \gamma_{3,2}^2}\right)\left(\frac{1}{s^2+\omega_2^2}\right), \end{aligned} \tag{24}$$

Therefore, ε is increased and k_1 is decreased in order to reduce the effect of $s_1(t)$. Also ε is increased and k_2 is decreased in order to reduce the effect of $s_2(t)$.

Fig. 3 and Fig. 4 are simulation results when $s_1(t) = \sin 3t, s_2(t) = 0$ and $s_1(t) = 0, s_2(t) = \sin 3t$, respectively.

It shows that the effect of $s_1(t)$ and $s_2(t)$ can be reduced if ε is a large value.

4.3 DC and AC measurement noise

Measurement noise of feedback signal is assumed as

DC and AC component in Eq. (20).

$$s_1(t) = \alpha_1 + \rho_1 \sin\omega_1, \quad s_2(t) = \alpha_2 + \rho_2 \sin\omega_2. \tag{25}$$

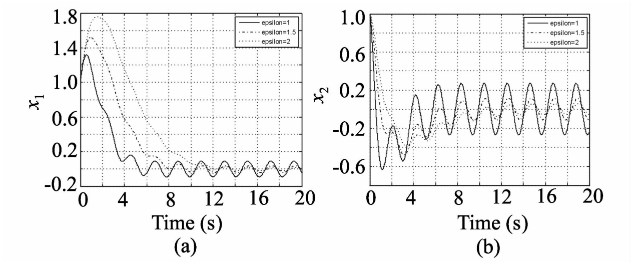

Fig. 3 Simulation result when $s_1(t) = \sin 3t, s_2(t) = 0$: (a) x_1 , (b) x_2

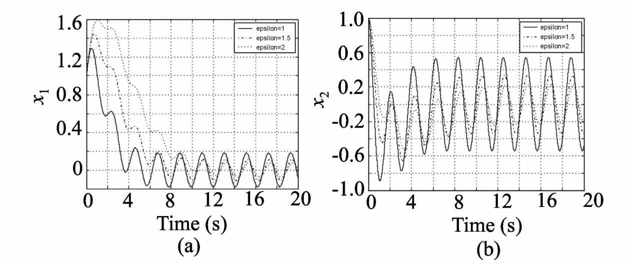

Fig. 4 Simulation result when $s_1(t) = 0, s_2(t) = \sin 3t$: (a) x_1 , (b) x_2

After applying Eq. (25) to Eq. (20), using the final value theorem the remaining term is

$$-\alpha_1 + \frac{\rho_{k1}}{\varepsilon^3}\left(\frac{\beta_{3,1}\omega_1 - \gamma_{3,1}s}{\beta_{3,1}^3 + \gamma_{3,1}^2}\right)\left(\frac{1}{s^2 + \omega_1^2}\right) - \frac{\alpha_2 k_2 \varepsilon}{k_1} + \frac{\rho_2 k_2}{\varepsilon^2}\left(\frac{\beta_{3,2}\omega_2 - \gamma_{3,2}s}{\beta_{3,2}^2 + \gamma_{3,2}^2}\right)\left(\frac{1}{s^2 + \omega_2^2}\right). \tag{26}$$

Therefore, k_1, k_2 are decreased in order to reduce the effect of both DC and AC measurement noise. However, DC component of $s_1(t)$ is not reduced.

Fig. 5 and Fig. 6 are simulation results when $s_1(t) = 1 + \sin 3t, s_2(t) = 0$ and $s_1(t) = 0, s_2(t) = 1 + \sin 3t$, respectively.

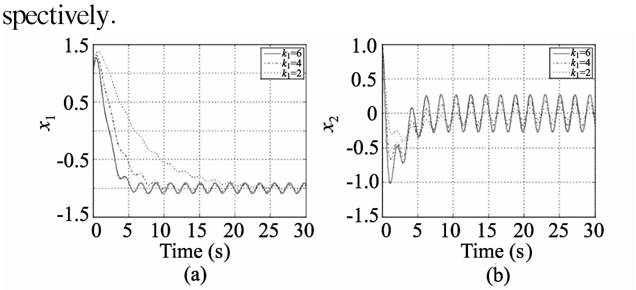

Fig. 5 Simulation result when $s_1(t) = 1 + \sin 3t, s_2(t) = 0$: (a) x_1 , (b) x_2

It shows that the effect of $s_1(t)$ and $s_2(t)$ can be reduced except for DC component of $s_1(t)$ if k_1, k_2 are smallvalues.

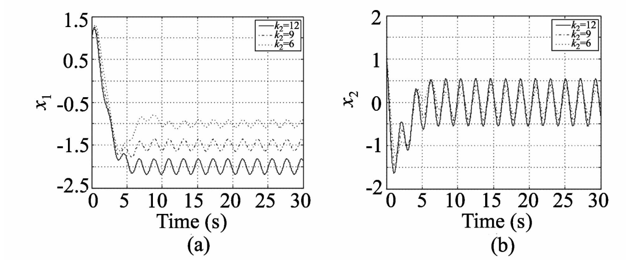

Fig. 6 Simulation result when $s_1(t) = 0, s_2(t) = 1 + \sin 3t$: (a) x_1 , (b) x_2

5 Conclusions

In this paper, we propose a gain-scaling controller to compensate measurement noise of feedback sensor. Controller equipped with compensator is introduced. Three types of measurement noise as DC, AC, and DC and AC component are assumed and system analysis using Laplace transform is shown. Gain-scaling factor ε should be decreased if measurement noise is DC component, ε should be increased if it is AC component, and the gain K should be decreased if it is DC and AC component. However, DC component of $s_1(t)$ cannot be compensated by using ε or K .

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