# Polynomial Root Finding on Frequency Estimation with Sub-Nyquist Temporal Sampling

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**Abstract** – This paper addresses an algebraic approach for wideband frequency estimation with sub-Nyquist temporal sampling. Firstly, an algorithm based on double polynomial root finding procedure to estimate aliasing frequencies and joint aliasing frequencies-time delay phases in multi-signal situation is presented. Since the sum of time delay phases determined from the least squares estimation shows the characteristics of the corresponding parameters pairs, then the pair-matching method is conducted by combining it with estimated parameters mentioned above. Although the proposed method is computationally simpler than the conventional schemes, simulation results show that it can approach optimum estimation performance.

***Key words*** – *wideband frequency estimation*; *sub-nyquist sampling*; *polynomial root finding*; *pair matching*

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Frequency estimation is an important problem in many fields including wireless communications, signal analysis systems and Electronic Warfare (EW). The most well-known methods are those based on Fast Fourier Transform (FFT) digital signal processor from which frequencies estimates are obtained by picking the peaks[1]. These methods provide for sample accuracy. If better accuracy is required, computationally interpolation is needed. For wideband application, if the signal frequency exceeds half the sampling rate, frequency aliasing will occur, leading to attendant problems of ambiguity. Another group of methods are time-delay and cross-spectrum methods[2]. These methods estimate the quantity $\exp(j\omega_m\tau)$ based on phase difference in the time domain, where $\omega_m$ is the $m$th frequency, $\tau$ is the delay time. The methods have low computation burden, but poor performance. Large errors can be resulted in due to the effects of noise perturbations. Their methods are not applicable to multiple frequencies.

In many systems, such as EW system, wideband receivers must have a wide bandwidth extending into the GHz region[3]. Although many high-speed A/D converters are available, low processing speed following the converters may limit the overall operation of the receiver. The problem of wideband high-frequency applications with sub-Nyquist sampling has been investigated for many years. In Ref. [4-5], frequency unambiguous estimation was achieved by using two or more A/D converters with different sampling rate. Based on Chinese remainder theorem, these sampling rates could satisfy the unambiguous frequency interval. In Ref. [6-7], eigen-subspace-based methods utilize an auxiliary time-delay channel to disambiguate the aliasing frequency , which can solve the problem of multi frequencies simultaneously . These approaches have excellent performance but more amount of computation.

In this paper, we propose a novel approach for wideband frequency estimation with sub-sampled temporal data by using polynomial root finding method. By utilizing the temporal data from sub-Nyquist sampling channels which are delayed with multiple delay time, two polynomial equations for estimating aliasing frequencies and joint aliasing frequencies-time delay phases can be obtained. Sequentially pair them by combining them with the sum of time delay phases determined from the least squares estimation. A closed-form solution for unambiguous frequency estimation is provided. As a result, the proposed approach is computationally simpler and comparable to the subspace-based methods in performance.

The rest of this paper is organized as follows. Section 2 gives a detailed description of the proposed method. In section 3, performance is evaluated for multi frequencies. Section 4 is the conclusion of this paper.

## 1 Frequency disambiguation

The mathematical proof of the proposed method is given as follows. The model of the noisy data is described by

$$x_0(n) = \sum_{k=1}^{K} s_k \exp(j2\pi f_k n/f_s) + \upsilon_{x_0}(n). \quad (1)$$

The signal $x_0(n)$ consists of $K$ sinusoids descried by the frequency $\{f_k\}$ and the complex amplitudes $\{s_k\}$, $(k = 1, \cdots, K)$. Let the sub-sampling rate be denoted as $f_s$ which does not exceed two

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times the smallest frequency. $M-1$ more processors can be used while each works with delay time $m\tau$ $(1 \leqslant m \leqslant M-1)$, where $\tau$ is the delay time sufficiently long to satisfy the frequency disambiguation requirement. The output of the processor can be written as

$$x_m(n) = \sum_{k=1}^{K} s_k \exp(j2\pi f_k(n/f_s + m\tau)) + \upsilon_{x_m}(n), \quad 0 \leqslant m \leqslant M-1, \tag{2}$$

where $\upsilon_{x_m}(n)$ is a zero-mean complex Gaussian white noise with variance $\sigma^2$. In the matrix form, the noisy data matrix formed by sample data $x_m(n)$ $(0 \leqslant m \leqslant M-1)$ can be expressed by $\boldsymbol{X} = [\boldsymbol{X}_0, \boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_{M-1}]$, where $\boldsymbol{X}_m = [x_m(n), x_m(n+1), \cdots, x_m(n+M-1)]^{\mathrm{T}}$ with T denotes the transpose operator. From the diagonal elements of the matrix, we have Vandermonde vector $\boldsymbol{Y} = [y_0(n), y_1(n), \cdots, y_{M-1}(n)]^{\mathrm{T}}$. where

$$y_m(n) = \sum_{k=1}^{K} s_k \exp(j2\pi f_k n/f_s + j2\pi f_k m(1/f_s + \tau)) + \upsilon_{y_m}(n), \quad \upsilon_{y_m}(n) = \upsilon_{x_m}(n+m). \tag{3}$$

In a similar way, elements of Vandermonde vector $\boldsymbol{Z} = [z_0(n), z_1(n), \cdots, z_{M-1}(n)]^{\mathrm{T}}$ formed by the first row of the matrix $\boldsymbol{X}$ can be written as

$$Z_m(n) = \sum_{k=1}^{K} s_k \exp(j2\pi f_k n/f_s + j2\pi f_k m\tau) + \upsilon_{z_m}(n), \quad \upsilon_{z_m}(n) = \upsilon_{x_m}(n). \tag{4}$$

It is shown in Appendix that elements of Vandermonde vectors $\boldsymbol{X}_0$ can be rewritten as

$$x_0(n+m) = \sum_{k=1}^{K} (-1)^{k+1} W_K^{(k)} x_0(n+m-k). \tag{5}$$

Elements of $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are also rewritten as

$$y_m(n) = \sum_{k=1}^{K} (-1)^{k+1} V_K^{(k)} y_{m-k}(n) \tag{6}$$

$$z_m(n) = \sum_{k=1}^{K} (-1)^{k+1} U_K^{(k)} z_{m-k}(n) \tag{7}$$

Matrix form of Eq. (5) can be expressed by

$$\begin{bmatrix} x_0(n+K) & -x_0(n+K-1) & \cdots(-1)^{K+1}x_0(n+1) \\ x_0(n+K+1) & -x_0(n+K) & \cdots(-1)^{K+1}x_0(n+2) \\ \cdots & & \\ x_0(n+2K-1) & -x_0(n+2K-2) & \cdots(-1)^{K+1}x_0(n+K) \end{bmatrix} \cdot \begin{bmatrix} W_K^{(1)} \\ W_K^{(2)} \\ \cdots \\ W_K^{(K)} \end{bmatrix} = \begin{bmatrix} x_0(n+K+1) \\ x_0(n+K+2) \\ \cdots \\ x_0(n+2K) \end{bmatrix}, \tag{8}$$

where

$$W_K^{(1)} = w_1 + w_2 + \cdots + w_K,$$
$$W_K^{(2)} = w_1 w_2 + w_1 w_3 + \cdots + w_{K-1} w_K,$$
$$\cdots$$
$$W_K^{(k)} = w_1 w_2 \cdots w_k + \cdots + w_{K-k+1} w_{K-k} \cdots w_K,$$
$$\cdots$$
$$W_K^{(K)} = w_1 w_2 \cdots w_K,$$
$$w_k = \exp(j2\pi f_k/f_s), \; k = 1, 2, \cdots, K.$$

From the principle of Vieta's theorem, we know that $w_k$ is the root of the below polynomial equation.

$$D(w) = w^K - W_1 w^{K-1} + \cdots + (-1)^{K-1} W_{K-1} w + (-1)^K W_k = 0. \tag{9}$$

The $K$ roots closest to the unit circle of the polynomial $D(w)$ allow to estimate the aliasing frequencies, given by $\hat{w}_k = \exp(j2\pi \hat{f}_k/f_s)$ $(k = 1, 2, \cdots, K)$. In equation form, this is equivalent to estimate joint aliasing frequency-time delay phases from Eq. (6), given by $\hat{\upsilon}_k = \exp(j2\pi \hat{f}_k(1/f_s + \tau))$ $(k = 1, 2, \cdots, K)$. More important is the data association problem wherein the aliasing frequency must be paired with the right joint aliasing frequency-time delay phase so that aliasing frequency can be disambiguated properly. The sum of time delay phases constructed by Eq. (7) shows the characteristics of the corresponding parameters pairs, which is satisfying $\hat{u} = \sum_{k=1}^{K} \exp(j2\pi \hat{f}_k \tau)$ and independent of sub-sampling rate $f_s$. The estimation of $\hat{u}$ is then determined from the Least Squares (LS) estimation of $U_K^{(1)}$, where $[U_K^{(1)}, U_K^{(2)}, \cdots, U_K^{(K)}]^{\mathrm{T}} = (\boldsymbol{Z}_K^{\mathrm{T}} \boldsymbol{Z} \boldsymbol{K})^{-1} \boldsymbol{Z}_K^{\mathrm{T}} \boldsymbol{z}_k$

$$\boldsymbol{Z}_K = \begin{bmatrix} z_K(n) & -z_{K-1}(n) & \cdots(-1)^{K+1}z_1(n) \\ z_{K+1}(n) & -z_K(n) & \cdots(-1)^{K+1}z_2(n) \\ \cdots & & \\ z_{2K-1}(n) & -z_{2K-2}(n) & \cdots(-1)^{K+1}z_K(n) \end{bmatrix},$$

$$\boldsymbol{z}_k = \begin{bmatrix} z_{K+1}(n) \\ z_{K+2}(n) \\ \cdots \\ Z_{2K}(n) \end{bmatrix}.$$

In addition, since $\hat{\upsilon}_k / \hat{w}_k = \exp(j2\pi f_k \tau)$, which is also independent of sub-sampling rate $f_s$, $\hat{w}_k$ and $\hat{\upsilon}_k$ can be paired properly via the formulation in Eq. (10).

$$(\hat{w}_k, \hat{\upsilon}_k) \in \underset{k \in [1,K]}{\arg\min} \left| \sum_{k_1, k_2 = 1}^{K} \frac{\hat{\upsilon}_{k_2}}{\hat{w}_{k_1}} - \hat{u} \right|. \tag{10}$$

The aliasing frequency of thesinusoid is ultimately determined from $\hat{w}_k$ is

$$f_k^{\text{fine}} - l_k f_s = \frac{f_s}{2\pi} \arg(\hat{w}_k), \tag{11}$$

where the true frequency $f_k$ is denoted as $f_k^{\text{fine}}$, $l_k$ is an integer and $\arg(x)$ is the angle of $x$. The ambiguity can be resolved with another paired estimated frequency denoted as $f_k^{\text{coarse}}$, which is obtained from $\hat{v}_k$ by

$$f_k^{\text{coarse}} = \arg\left(\frac{\hat{\upsilon}_k}{\hat{w}_k}\right) / (2\pi\tau), \tag{12}$$

where sinceis $\hat{v}_k / \hat{w}_k$ is independent of $f_s$, $f_k^{\text{coarse}}$ is the unambiguous frequency estimation, but has poor performance compared to $f_k^{\text{fine}}$ due to the effect of noise and delay time perturbation. Equating the expressions for $f_k^{\text{fine}}$ and $f_k^{\text{coarse}}$, the estimation of $l_k$ is

$$l_k = \arg\min_{l_k} \left| \arg\left(\frac{\hat{v}_k}{\hat{w}_k}\right)/(2\pi\tau) - \frac{f_s}{2\pi}\arg(\hat{w}_k) - l_k f_s \right|,$$
$$0 \leqslant l_k \leqslant floor\left(\frac{f_{\max}}{f_s}\right), \quad (13)$$

where $floor(x)$ is the integer closest to, but less than $x$, $f_{\max}$ is the upper limit on processible frequency.

On one hand, regarding major computational complexity, the number of multiplications for calculating $\hat{w}_k$, $\hat{v}_k$ and $\hat{u}$ includes $O(K^2)$ in the $W_K^k$, $V_K^k$, or $U_K^k(k=1,2,\cdots,K)$ computation and $O(K^3)$ in the polynomial rooting procedure. On the other hand, the eigen-subspace-based method involves $O(M^2)$ for cross-covariance computation and $O(M^3)$ for the eigen-value decomposition of the covariance matrix. For the typical conditions of $M>K$, the computational attractiveness of our proposed method is indicated.

## 2 Simulation results

In this section, signal frequency based on the polynomial root finding approach is estimated by simulation modeling and the performance as a function of the SNR is also evaluated. Two equal-strength signal sources in the interval [300 MHz, 500 MHz] are assumed throughout the tests. The sub-sampling rate $f_s$ is selected as 100 MHz. The tests below are all repeated for 500 times.

First, we evaluate the probability of failure pairing $P_f$ between $\hat{w}_k$ and $\hat{v}_k$ for values of $\sigma^2$ covering the range [−30 dB, −10 dB] and for selected values of delay time $\tau$ as shown in Fig. 1.

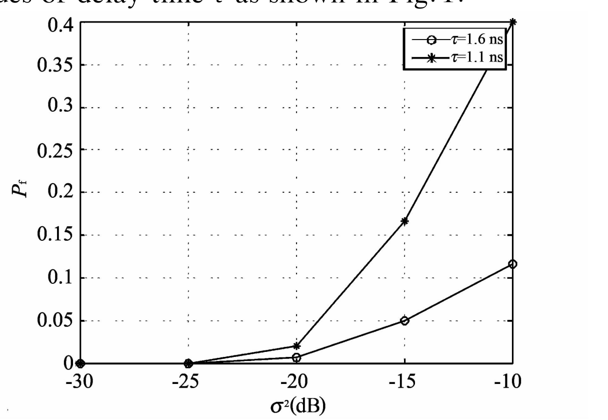


**Fig. 1** Probability of failure pairing $P_f$ versus $\sigma^2$ for selected values of delay time $\tau$

The number of temporal samples is set to 20. The performance degrades asdecreases while $\sigma^2$ is kept constant. It is because the smaller the delay time is becoming, the more severely the sum of right parameters pairs $\sum_{k=1}^{K}\frac{\hat{v}_k}{\hat{w}_k}$ is influenced by noise perturbations, which makes the incorrect pairing more probably. The figure emphasizes the fact that the probability of failure pairing depends on delay time. In the second example, the probability of disambiguating the aliasing frequencies properly $P_r$ versus SNR is investigated. For aliasing frequencies resolved with another paired set of frequencies which are related to delay- time, the disambiguation performance upgrades as delay time increases as shown in Fig. 2.

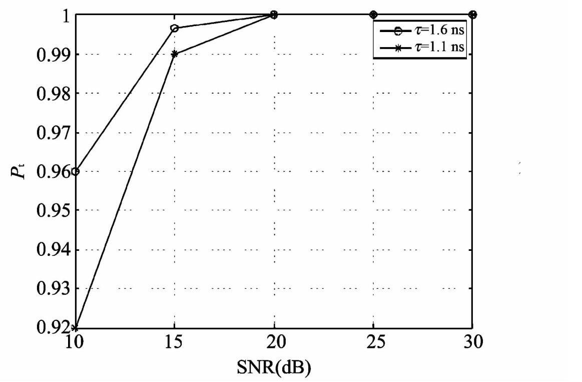


**Fig. 2** Probability of proper disambiguation $P_t$ versus SNR for selected values of delay time $\tau$

Fig. 2 reveals that the delay time should be set sufficiently long but to satisfy the frequency disambiguation requirement. After properly pairing $(\hat{w}_k, \hat{v}_k)k=1,2$ and disambiguating aliasing frequencies, the RMSE of frequency estimation versus SNR is studied in the final example. Fig. 3 shows that the estimation accuracy of the proposed algorithm is comparable to that of ESPRIT-based method.

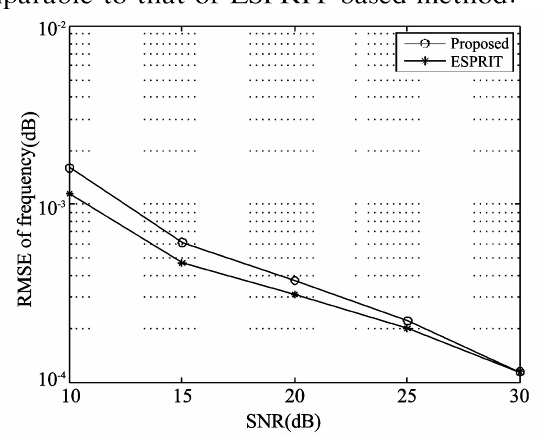


**Fig. 3** RMSE of frequency versus SNR

## 3 Conclusions

An algorithm that disambiguates the aliasing frequencies for wideband high-frequency applications is presented. The key idea in the algorithm development is to utilize the properly paired roots of the two polynomial equations constructed from sub-sampled temporal data. The performance of the method is studied in terms of the probability of failure due to noise. The effects of other system parameters on the performance of the proposed method are also investigated.

**Appendix A**

In this appendix, Eq. (5) is proven by finite induction. Let $x_0$ be rewritten as $x^{(N)}$. The subscript $N$ denotes number of signals.

1) The result clear holds for $N=1$, since

$$x^{(1)}(n+m) = W_1^{(1)} x^{(1)}(n+m-1) = w_1 x^{(1)}(n+m-1). \tag{14}$$

2) Consider $N=2$, assume $W_2^{(1)} = w_1 + w_2$ and $W_2^{(2)} = w_1 w_2$, then

$$x^{(2)}(n+m) = w_1^{n+m} s_1 + w_2^{n+m} s_2 = (w_1+w_2)(w_1^{n+m-1}s_1 + w_2^{n+m-1}s_2) - w_1 w_2 (w_1^{n+m-2}s_1 + w_2^{n+m-2}s_2) = w_2^{(1)} x^{(2)}(n+m-1) - W_2^{(2)} x^{(2)}(n+m-2). \tag{15}$$

3) Assume that Eq. (5) is valid for $N=K-1$, then

$$x^{(K-1)}(n+m) = \sum_{k=1}^{K-1} (-1)^{k+1} W_{K-1}^{(k)} x^{(K-1)}(n+m-k). \tag{16}$$

4) For $N=K$, the measurement equation is given by

$$x^{(K)}(n+m) = \sum_{k=1}^{K} (-1)^{k+1} W_K^{(k)} x^{(K)}(n+m-K) = [x^{(K)}(n+m-1), \cdots, (-1)^{k+1} x^{(K)}(n+m-k), \cdots, (-1)^{K+1} x^{(K)}(n+m-K)] \begin{bmatrix} w_K, 1, \cdots, 0, 0 \\ 0, w_K, 1, \cdots, 0 \\ \cdots \\ 0, 0, \cdots, 0, w_K \end{bmatrix} \begin{bmatrix} 1 \\ W_{K-1}^{(1)} \\ W_{K-1}^{(2)} \\ \cdots \\ W_{K-1}^{(K-1)} \end{bmatrix} =$$

$$\begin{bmatrix} x^{(K)}(n+m-1) w_K \\ x^{(K)}(n+m-1) - x^{(K)}(n+m-2) w_K \\ \cdots \\ (-1)^k x^{(K)}(n+m-k+1) + (-1)^{k+1} x^{(K)}(n+m-k) w_K \\ \cdots \\ (-1)^K x^{(K)}(n+m-K+1) + (-1)^{K+1} x^{(K)}(n+m-K) w_K \end{bmatrix} \begin{bmatrix} 1 \\ W_{K-1}^{(1)} \\ W_{K-1}^{(2)} \\ \cdots \\ W_{K-1}^{(K-1)} \end{bmatrix}, \tag{17}$$

$$\begin{aligned} &(-1)^k x^{(K)}(n+m-k+1) + (-1)^{k+1} x^{(K)}(n+m-k) w_K \\ &= (-1)^k [(x^{(K-1)}(n+m-k+1) + s_K w_K^{n+m-k}) - (x^{(K-1)}(n+m-k) + s_K w_K^{n+m-k-1}) W_K] \\ &= (-1)^k (x^{(K-1)}(n+m-k+1) - x^{(K-1)}(n+m-k) w_K). \end{aligned} \tag{18}$$

Substituting Eq. (18) into Eq. (17) yields the result Eq. (19).

$$\begin{aligned} x^{(K)}(n+m) &= x^{(K)}(n+m-1) w_k + (x^{(K-1)}(n+m-1) - x^{(K-1)}(n+m-2) w_K) W_{K-1}^{(1)} + \cdots + (-1)^k (x^{(K-1)}(n+m-k+1) - x^{(K-1)}(n+m-k) w_K) W_{K-1}^{(k)} + \cdots + (-1)^K (x^{(K-1)}(n+m-K+1) - x^{(K-1)}(n+m-K) w_K) W_{K-1}^{(K-1)} \\ &= x^{(K)}(n+m-1) w_K + \sum_{k=1}^{K-1} (-1)^{k+1} W_{K-1}^{(k)} x^{(K-1)}(n+m-k) - w_k \sum_{k=1}^{K-1} (-1)^{k+1} W_{K-1}^{(k)} x^{(K-1)}(n+m-k-1) \\ &= x^{(K)}(n+m-1) w_k + x^{(K-1)}(n+m) - x^{(K-1)}(n+m-1) w_K \\ &= (x^{(K-1)}(n+m-1) + s_K w_K^{n+m-2}) w_K + x^{(K-1)}(n+m) - x^{(K-1)}(n+m-1) w_K \\ &= x^{(K-1)}(n+m) + s_K w_K^{n+m-1}. \end{aligned} \tag{19}$$

As a result, the equation comes as desired. This completes the proof of the proposition.

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