Comparison of Linearized Kalman Filter and Extended Kalman Filter for Satellite Motion States Estimation

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Abstract – The performance of the conventional Kalman filter depends on process and measurement noise statistics given by the system model and measurements. The conventional Kalman filter is usually used for a linear system, but it should not be used for estimating the state of a nonlinear system such as a satellite motion because it is difficult to obtain the desired estimation results. The linearized Kalman filtering approach and the extended Kalman filtering approach have been proposed for a general nonlinear system. The equations of satellite motion are described. The satellite motion states are estimated, and the relevant estimation errors are calculated through the estimation algorithms of the both above mentioned approaches implemented in Matlab are estimated. The performances of the extended Kalman filter and the linearized Kalman filter are compared. The simulation results show that the extended Kalman filter is much better than the linearized Kalman filter at the aspect of estimation effect.

Key words* – *nonlinear filtering approach*; *nonlinear system*; *satellite orbit*; *state space*; *state estimation

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1 Introduction

Since the Kalman filter[1,2] appears, it is usually used for estimating linear systems, but absolute linear systems do not really exist. All systems are ultimately nonlinear. Many systems are close enough to linear that linear estimation approaches give satisfactory results, but for some nonlinear systems the linear approaches for estimation no longer give good results. Therefore, many nonlinear estimation approaches are studied, such as Extended Kalman Filter (EKF)[3,4], Unscented Kalman Filter (UKF)[5-7], Particle Filter (PF)[8,9] and their combined approaches.

The satellite motion states determination problem consists of two basic parts: propagating the state estimates forward in time and updating the state estimate based upon the new measurements of the parameters which are functions of the states.

A simple example is taken for the linearized Kalman filter and the extended Kalman filter to develop the filtering approaches for a nonlinear system such as satellite motion states description. In the next section the updated equations are described corresponding to the linearized Kalman filter and the extended Kalman filter for a general nonlinear system. In the third section the nonlinear motion equation of a satellite is presented in the state space. In the fourth section the satellite motion states are estimated through both nonlinear Kalman filtering approaches. Finally, the performance of the extended Kalman filter against the linearized Kalman filter is demonstrated, and the conclusions are reached.

2 Nonlinear system model

Consider a general nonlinear system model and measurement model as

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= f(\boldsymbol{x}(t), t) + \boldsymbol{w}(t), \\ \boldsymbol{z}_k &= h_k(\boldsymbol{x}_k(t_k)) + \boldsymbol{v}_k, \end{aligned} \tag{1}$$

Where $\boldsymbol{x}(t)$ is the state vector of the system, $f(\boldsymbol{x}(t), t)$ is the process model, $\boldsymbol{w}(t)$ is the process noise assumed to be white Guassian with zero mean and covariance $\boldsymbol{Q}(t)$, $\boldsymbol{w}(t) \sim N(0, \boldsymbol{Q}(t))$, $\boldsymbol{z}_k$ is the measurement vector of the system at the time step k, $k = 1, 2, \cdots, h_k(\boldsymbol{x}_k(t_k))$ is the measurement model, v_k is the measurement noise assumed to be white Guassian with zero mean and covariance $\boldsymbol{R}_k$, $\boldsymbol{v}_k \sim N(0, \boldsymbol{R}_k)$, and assume that the process noise and the measurement noise are uncorrelated, $E[\boldsymbol{w}(t)\boldsymbol{v}_k^{\mathrm{T}}] = 0$ for all k and t.

The initial condition is assumed as

$$\begin{aligned} \boldsymbol{x}(0) &\sim N(\hat{x}_0, P_0), \\ P_0 &= E[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^{\mathrm{T}}]. \end{aligned}$$

3 Nonlinear filtering method

This section presents the linearized Kalman filter and the extended Kalman filter and gives their

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updated equations for estimation[10,11].

3.1 Linearized Kalman filtering method

For the nonlinear system model Eq. (1), the state estimate equation can be written as

$$\dot{\hat{x}}(t) = f(\bar{x}(t),t) + F(\bar{x}(t),t)[\hat{x}(t) - \bar{x}(t)], \quad (2)$$

where $f(\bar{x}(t),t)$ is the nonlinear function depending upon $\bar{x}(t)$ and t, $F(\bar{x}(t),t) = \left.\frac{\partial f(\boldsymbol{x}(t),t)}{\partial \boldsymbol{x}(t)}\right|_{x(t)=\bar{x}(t)}$ is the transition matrix, $\bar{x}(t)$ is the mean of the state vector $\boldsymbol{x}(t)$, $\hat{x}(t)$ is the estimation of the state vector $\boldsymbol{x}(t)$.

The error covariance equation is given as

$$\dot{\boldsymbol{P}}(t) = \boldsymbol{F}(\bar{x}(t),t)\boldsymbol{P}(t) + \boldsymbol{P}(t)\boldsymbol{F}^{\mathrm{T}}(\bar{x}(t),t) + \boldsymbol{Q}(t), \quad (3)$$

where $\boldsymbol{P}(t)$ is the covariance matrix.

The updated state estimate equation is given as

$$\hat{\boldsymbol{x}}_{k^+} = \hat{\boldsymbol{x}}_{k^-} + \boldsymbol{K}_k\{\boldsymbol{z}_k - h_k(\bar{x}(t_k)) - \boldsymbol{H}_k(\bar{x}(t_k))[\hat{\boldsymbol{x}}_{k^-}\bar{x}(t_k)]\}, \quad (4)$$

where $\boldsymbol{H}_k(\bar{x}(t_k)) = \left.\frac{\partial h_k(\boldsymbol{x}(t_k))}{\partial \boldsymbol{x}(t_k)}\right|_{x(t_k)=\bar{x}(t_k)}$ is the measurement matrix, $\boldsymbol{K}_k$ is the gain matrix.

$$\boldsymbol{K}_k = \boldsymbol{P}_{k^-}\boldsymbol{H}_k^{\mathrm{T}}(\bar{x}(t_k))[\boldsymbol{H}_k(\bar{x}(t_k))\boldsymbol{P}_{k^-}\boldsymbol{H}_k^{\mathrm{T}}(\bar{x}(t_k)) + \boldsymbol{R}_k]^{-1}. \quad (5)$$

The error covariance equation is given as

$$\boldsymbol{P}_{k^+} = [\boldsymbol{I} - \boldsymbol{K}_k\boldsymbol{H}_k(\bar{x}(t_k))]\boldsymbol{P}_{k^-}. \quad (6)$$

Eq. (2) ~ (6) constitute the linearized Kalman filtering algorithm for nonlinear systems with discrete measurements.

3.2 Extended Kalman filtering method

For the nonlinear system model Eq. (1), the state estimate propagation equation can be written as

$$\dot{\hat{\boldsymbol{x}}} = f(\hat{\boldsymbol{x}}(t),t), \quad (7)$$

where $\hat{x}(t)$ is the estimation of the state vector $\boldsymbol{x}(t)$, $\boldsymbol{f}(\hat{\boldsymbol{x}}(t),t)$ is the nonlinear function depending upon $\hat{\boldsymbol{x}}(t)$ and t.

The error covariance propagation equation is given as

$$\dot{P}(t) = \boldsymbol{F}(\hat{\boldsymbol{x}}(t),t)\boldsymbol{P}(t) + \boldsymbol{P}(t)\boldsymbol{F}^{\mathrm{T}}(\hat{\boldsymbol{x}}(t),t) + \boldsymbol{Q}(t), \quad (8)$$

where $\boldsymbol{F}(\hat{\boldsymbol{x}}(t),t) = \left.\frac{\partial f(\boldsymbol{x}(t),t)}{\partial \boldsymbol{x}(t)}\right|_{x(t)=\hat{x}(t)}$ is the transition matrix, $\boldsymbol{P}(t)$ is estimation error covariance matrix.

The updated state estimate equation is given as

$$\hat{\boldsymbol{x}}_{k^+} = \hat{\boldsymbol{x}}_{k^-} + \boldsymbol{K}_k[\boldsymbol{z}_k - h_k(\hat{\boldsymbol{x}}(t_{k^-}))], \quad (9)$$

where $\boldsymbol{K}_k$ is the gain matrix

$$\boldsymbol{K}_k = \boldsymbol{P}_{k^-}\boldsymbol{H}_k^{\mathrm{T}}(\hat{\boldsymbol{x}}(t_{k^-}))[\boldsymbol{H}_k(\hat{\boldsymbol{x}}(t_{k^-}))\boldsymbol{P}_{k^-}\boldsymbol{H}_k^{\mathrm{T}}(\hat{\boldsymbol{x}}(t_{k^-})) + \boldsymbol{R}_k]^{-1}, \quad (10)$$

where $\boldsymbol{H}_k(\hat{\boldsymbol{x}}(t_{k^-})) = \left.\frac{\partial h_k(\boldsymbol{x}(t_k))}{\partial \boldsymbol{x}(t_k)}\right|_{x(t_k)=\hat{x}(t_{k^-})}$ is the measurement matrix.

The updated error covariance equation is given as

$$\boldsymbol{P}_{k^+} = [\boldsymbol{I} - \boldsymbol{K}_k\boldsymbol{H}_k(\hat{\boldsymbol{x}}(t_{k^-}))]\boldsymbol{P}_{k^-}. \quad (11)$$

Eq. (7) ~ (11) constitute the extended Kalman filtering algorithm for nonlinear systems with discrete measurements.

4 Satellite motion equation

A planar model for a satellite orbiting around the earth can be described as[12]

$$\ddot{r} = r\dot{\theta}^2 - \frac{GM}{r^2} + 2,$$
$$\ddot{\theta} = -\frac{2\dot{\theta}\dot{r}}{r}, \quad (12)$$

where $\ddot{r}$ is the eistance of the satellite from the center of the earth, θ is the angular position of the satellite in its orbit, G is the universal gravitational constant, M is the mass of the earth, and w is random noise due to space debris, atmospheric drag, outgassing, and so on.

The state variables are chosen as

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \end{bmatrix}. \quad (13)$$

The description of the state-space is

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \dot{r} \\ \ddot{r} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} + \boldsymbol{w} = \begin{bmatrix} \dot{r} \\ r\dot{\theta}^2 - \frac{GM}{r^2} \\ \dot{\theta} \\ -\frac{2\dot{\theta}\dot{r}}{r} \end{bmatrix} + \boldsymbol{w}$$
$$= \begin{bmatrix} \dot{\boldsymbol{x}}_2 \\ \boldsymbol{x}_1\boldsymbol{x}_4^2 - \frac{\boldsymbol{GM}}{\boldsymbol{x}_1^2} \\ \dot{\boldsymbol{x}}_4 \\ -\frac{2\boldsymbol{x}_4\boldsymbol{x}_2}{\boldsymbol{x}_1} \end{bmatrix} + \boldsymbol{w}. \quad (14)$$

The measurement equation is

$$\boldsymbol{z} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \end{bmatrix} + \boldsymbol{v} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} + \boldsymbol{v}. \quad (15)$$

5 Simulation

The linearized system matrix is given as

$$\boldsymbol{F} = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ x_4^2 + 2GM/x_1^3 & 0 & 0 & 2x_1x_4 \\ 0 & 0 & 0 & 1 \\ 2x_4x_2/x_1^2 & -2x_4x_1 & 0 & -2x_2/x_1 \end{bmatrix}, \quad (16)$$

$$\boldsymbol{H} = \frac{\partial h}{\partial x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{17}$$

where the dependence of x upon t, and f upon x and t, and h upon x and t, is suppressed for notational convenience.

In order for the orbit to have a constant radius when $\boldsymbol{w}=0$, then $\ddot{r} = r\dot{\theta} - GM/r^2$, which means $\dot{\theta} = \sqrt{GM/r^3}$, i. e. $x_4 = \sqrt{GM/r^3} = \sqrt{GM/x_1^3}$ or $GM = x_4^2 x_1^3$.

Note that at the linearization point $\dot{r}=0$, i. e. $GM = r_0^3 \omega_0^2$, then the matrix $\boldsymbol{F}$ at the linearization point is given as

$$\boldsymbol{F} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega_0^2 & 0 & 0 & 2r_0\omega_0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega_0 r_0 & 0 & 0 \end{bmatrix}. \tag{18}$$

Set the initial conditions as follows

$$G = 6.674\,2 \times 10^{-11}\ \mathrm{m^3/kg/s^2},$$
$$M = 5.98 \times 10^{24}\ \mathrm{kg},$$
$$\boldsymbol{w} \sim N(0, 10^{-6}),$$
$$r = r_0 = 6.57 \times 10^{-6}\ \mathrm{m},\ \dot{r} = 0,$$
$$\theta = \omega_0 T,\ \dot{\theta} = \omega_0 = \sqrt{\frac{GM}{r_0^3}},$$
$$\boldsymbol{x}(0) = [r_0 \quad 0 \quad 0 \quad 1.05\omega_0]^{\mathrm{T}},$$
$$\hat{\boldsymbol{x}}(0) = \boldsymbol{x}(0),\ \boldsymbol{P}(0) = \mathrm{diag}(0,0,0,0).$$

Using Matlab programs, the estimation results of both the linearized Kalman filter and extended Kalman filter can be obtained as shown in Fig. 1. The curves in Fig. 1 briefly show that the estimation values of the extended Kalman filter are much better than these of the linearized Kalman filter to approach the true values.

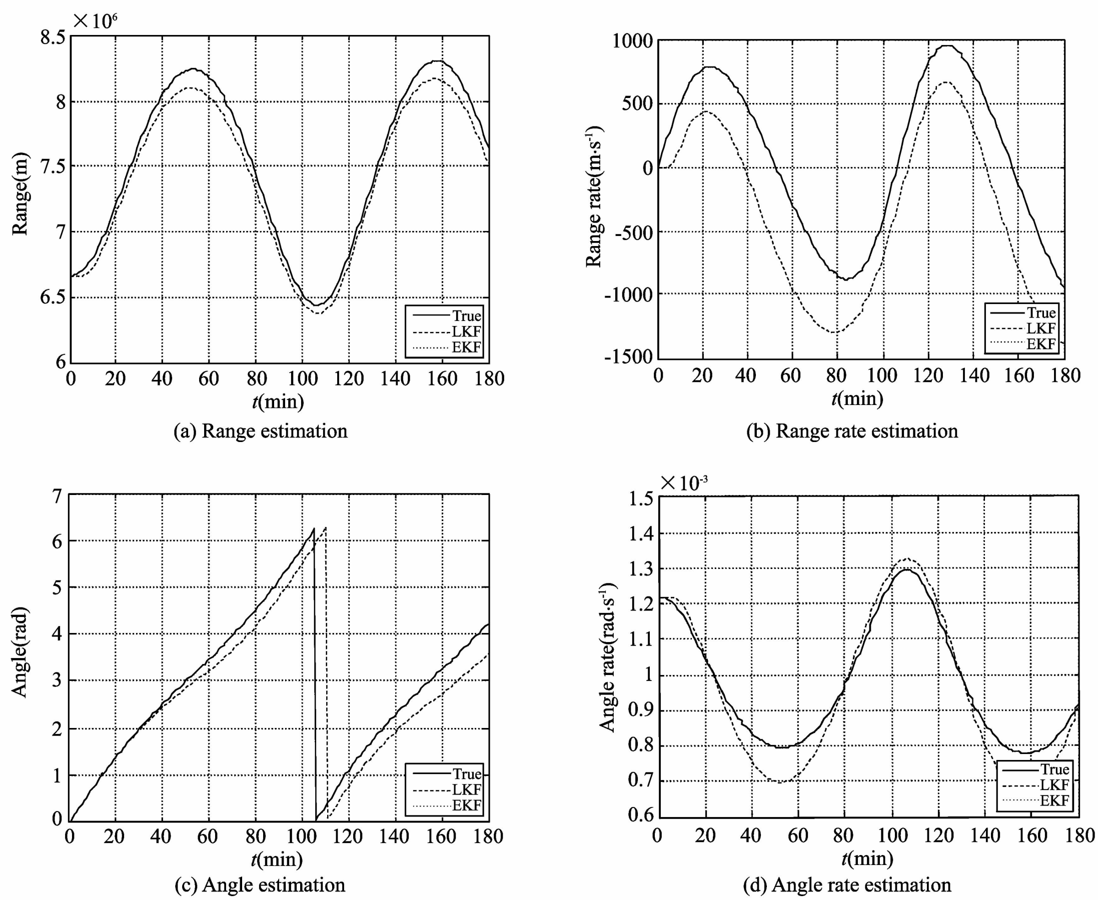
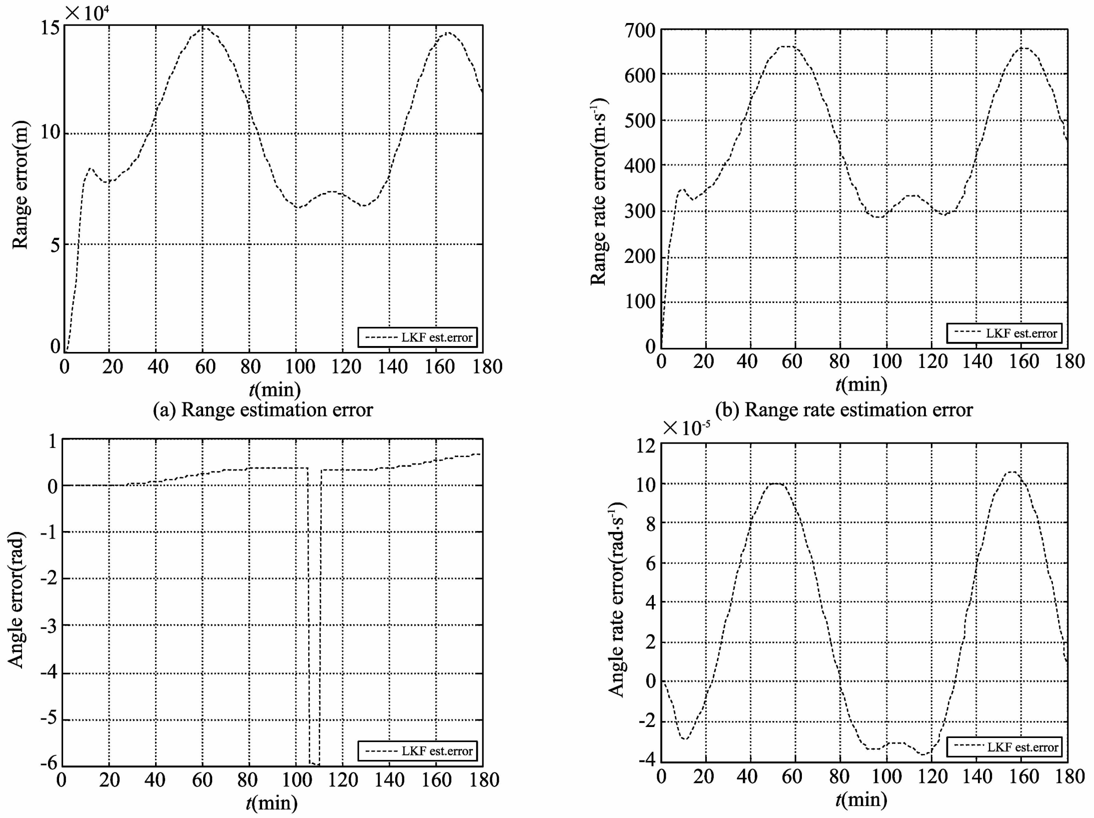


Fig. 1 The comparison between estimation results obtained by linearized Kalman filter and extended Kalman filter

The estimation errors of the linearized Kalman filter are shown in Fig. 2.

The estimation errors of the extended Kalman filter are shown in Fig. 3.

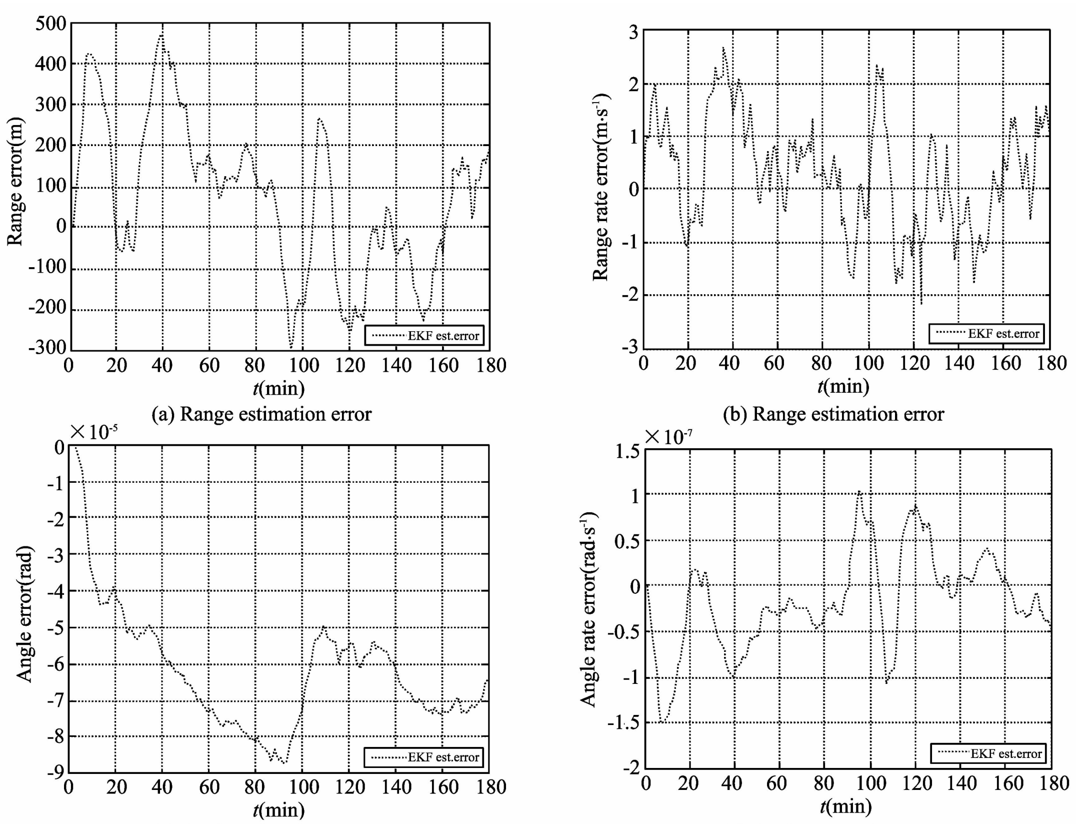
The comparism of the curves in Fig. 2 and Fig. 3 shows that the estimation errors of the extended Kalman filter are fewer than the these of the linearized Kalman filter. The reason is that $\bar{\boldsymbol{x}}(t)$ of the linearized Kalman filtering algorithm is usually not as close to the true trajectory as $\hat{\boldsymbol{x}}(t)$ is of the extended Kalman filtering algorithm. So, it shows the performance of the extended Kalman filter is much better than that of the linearized Kalman filter at the aspect of satellite motion states estimation.



(a) Range estimation error (b) Range rate estimation error

(c) Angle estimation error (d) Angle rate estimation error

Fig. 2 Estimation errors of the linearized Kalman filter



(a) Range estimation error (b) Range estimation error

(c) Angle estimation error (d) Angle rate estimation error

Fig. 3 Estimation errors of the extended Kalman filter

6 Conclusions

The extended Kalman filtering approach can be used for state estimation of a nonlinear system such as satellite motion states description. The simulation results demonstrat that extended Kalman filtering algorithm is much better than the linearized Kalman filter at aspect of satellite motion states estimation.

As it is known that there are many nonlinear filtering approaches which are better than the extended Kalman filtering approach for the satellite motion states estimation, the other approaches, such as UKF, PF, will be discussed in further studies.

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