H_∞ Control Based on LMIs for a Class of Time-delay Switched Systems

Chun-ming LI(李春明), Xue-min TIAN(田学民)
(*College of Information and Control Engineering, China University of Petroleum, Dongying 257061, China*)

Abstract – **The problem of H_∞ stability analysis and control synthesis of switched systems with delayed states under arbitrary switching laws is considered. By means of Lyapunov function and linear matrix inequality tools, sufficient condition of H_∞ stability is presented in terms of linear matrix inequalities. Furthermore, the robust H_∞ control synthesis via state feedback and output feedback is studied. Finally, a numerical example is given to demonstrate the effectiveness of the proposed method.**

Key words – ***time-delay switched systems; H_∞ performance; Lyapunov function; linear matrix inequality***

Manuscript Number: 1674-8042(2010)03-0293-04

dio: 10.3969/j.issn.1674-8042.2010.03.20

1 Introduction

Switched systems are an important class of hybrid systems consisting of a family of subsystems and a switching law that specifies which subsystem will be activated along the system trajectory at each instant of time[1,2]. During the last decade, switched systems and switching control have gained much attention because this field is not only of practical importance, but also academically challenging. Many real-world processes and systems, including chemical processes, computer controlled systems, switched circuits, and so on. Moreover, switched systems find considerable applications in many other engineering fields[3-5].

As is well known, time delays are very common phenomena in many real control systems which are a great source of instability and poor performance. During the past decades, the stability of control systems with time delay has received considerable attention. However, few existent results for switched systems have considered the factor of time delay. Ref. [6-8] established the necessary and sufficient criteria for controllability and reachability of switched linear systems with delayed control input. Ref. [9] studied the stability and L_2 gain analysis for switched linear systems with time delay. On the other hand, H_∞ control is one of the most active subfields of research and H_∞ performance is also an extremely important performance. Unfortunately few results are concerned with H_∞ control of switched systems. Ref. [10] and [11] have studied disturbance attenuation of switched systems. Several other papers such as Ref. [12-14] were also dedicated to the study of H_∞ related problems for some kind of hybrid systems. Ref. [15] has studied robust H_∞ control of switched linear systems with norm-bounded time-varying uncertainties by using multiple Lyapunov functions method. Ref. [16] deals with the stabilization and robust H_∞ control of discrete switched system with time delay based on the average dwell time method.

This paper studies H_∞ stability analysis and control synthesis of continuous-time switched linear systems with delayed states by linear matrix inequality approach together with Lyapunov function method. Firstly, by constructing linear matrix inequalities via Schur complement formula, a sufficient condition of H_∞ stability is presented in terms of matrix inequality. Next, sufficient condition of H_∞ control is given under state feedback and output feedback. The results are formulated as linear matrix inequality conditions which can be partly considered as extension of existing results for linear time-invariant time-delay systems and partly considered as extention of existing results for switched linear systems without time delay.

The paper is organized as follows. In Section 2 we present preliminaries including system description, assumption and definition. Main results are presented in Section 3. A numerical example is included in Section 4. Finally, Section 5 briefly concludes the work.

2 Preliminaries

We consider the following switched systems with time delay:

$$\begin{aligned} \dot{x}(t) &= \boldsymbol{A}_\sigma x(t) + \boldsymbol{A}_{d\sigma} x(t-d) + \boldsymbol{B}_{1\sigma}\omega(t) + \boldsymbol{B}_{2\sigma}u(t), \\ z(t) &= \boldsymbol{C}_{1\sigma}x(t) + \boldsymbol{D}_\sigma \omega(t), \\ y(t) &= \boldsymbol{C}_{2\sigma}x(t), \\ x(t) &= \varphi(t) \quad t \in [-d, 0]. \end{aligned} \tag{1}$$

※ **Received:** 2010-06-18
Project supported: This work was supported by the National "863" Foundation of China under Grant 2007AA04Z193
Corresponding author: Chun-ming LI(chunming-li@163.com)

where $x(t) \in R^n$ is the state; $u(t) \in R^k$ is the control input; $\omega(t) \in R^m$ is the exogenous disturbance input which belongs to $L_2[0,\infty]$; $z(t) \in R^P$ is controlled output; $y(t) \in R^j$ is the measurement output; $\varphi(\cdot)$ is a initial function on $[-d,0]$ and $d \geqslant 0$ is time delay of state; σ: $Z_{0+} \to M = \{1,2,\cdots,m\}$ is the switching signal, moreover, $\sigma = 1$ means that the i th subsystems is activated; $\boldsymbol{A}_i, \boldsymbol{A}_{di}, \boldsymbol{B}_{1i}, \boldsymbol{B}_{2i}, \boldsymbol{C}_{1i}, \boldsymbol{C}_{2i}$ and $\boldsymbol{D}_i$ are constant matrices of appropriate dimensions.

Without loss of generally, we make the following assumption.

Assumption: $\boldsymbol{C}_{2i}(\forall \in M)$ are all full row rank matrices. Let $u = K_i x$, then the closed-loop system of Eq. (1) is

$$\dot{x}(t) = (\boldsymbol{A}_i + \boldsymbol{B}_{2i}K_i)x(t) + \boldsymbol{A}_{di}x(t-d) + \boldsymbol{B}_{1i}\omega(t),$$
$$z(t) = \boldsymbol{C}_{1i}x(t) + \boldsymbol{D}_i\omega(t). \tag{2}$$

Applying output feedback control $u = F_i y$, we get the closed-loop system

$$\dot{x}(t) = (\boldsymbol{A}_i + \boldsymbol{B}_{2i}K_iC_{2i})x(t) + \boldsymbol{A}_{di}x(t-d) + \boldsymbol{B}_{1i}\omega(t),$$
$$z(t) = \boldsymbol{C}_{1i}x(t) + \boldsymbol{D}_i\omega(t). \tag{3}$$

We first consider the non-input system of

$$\dot{x}(t) = \boldsymbol{A}_\sigma x(t) + \boldsymbol{A}_{d\sigma}x(t-d) + \boldsymbol{B}_{1\sigma}\omega(t),$$
$$z(t) = \boldsymbol{C}_{1\sigma}x(t) + \boldsymbol{D}_\sigma\omega(t). \tag{4}$$

Definition: Given a constant $\gamma > 0$, the switched system (4) is said to be stabilizable with H_∞ disturbance attenuation γ via switching if there exists a switching rule σ such that under this switching, it satisfies

1) System (4) with $\omega = 0$ is globally asymptotically stabilizable.

2) With zero-initial condition $x(0)=0$, $\| z(t) \|_2 < \gamma \| \omega(t) \|_2$ holds for all nonzero exogenous disturbance input $\omega(t) \in L_2[0,\infty]$.

3 The main results

Theorem 1: Given a constant $\gamma > 0$, the switched system (4) is stabilizable with H_∞ disturbance attenuation γ for any arbitrary switching sequence if there exist symmetric matrices $\boldsymbol{P} > 0$ and $\boldsymbol{S} > 0$ such that the following matrix inequalities

$$\begin{bmatrix} A_i^T P + PA_i + S & PB_{1i} & C_{1i}^T & PA_{di} \\ B_{1i}^T P & -\gamma I & D_i^T & 0 \\ C_{1i} & D_i & -\gamma I & 0 \\ A_{di}^T P & 0 & 0 & -S \end{bmatrix} < 0,\ i \in M, \tag{5}$$

are satisfied.

Proof: If there exist symmetric positive matrices P and S such that Eq. (5) holds. We construct the following Lyapunov function

$$V(x) = x^T(t)Px(t) + \int_{i-d}^{i} x^T(\lambda)Sx(\lambda)d\lambda,$$

Then, $V(x)$ is positive-definitive. The time derivative of $V(x)$ along the trajectory of Eq. (4) with $\omega(t) = 0$ is

$$\dot{V}(t) = \dot{x}^T(t)Px(t) + ^T(t)P\dot{x}(t) + x^T(t)Sx(t) - x^T(t-d)Sx(t-d) =$$
$$\begin{bmatrix} x(t) \\ x(t-d) \end{bmatrix}^T \begin{bmatrix} A_i^T P + PA_i + S & PA_{di} \\ A_{di}^T P & -S \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-d) \end{bmatrix}.$$

It can be easily verified from Eq. (5) that $\dot{V}(x) < 0$ holds. Therefore, $V(x)$ decreases along solutions of system (4), which implies asymptotic stability.

In the following, we verify disturbance attenuation property. For this sake, let us introduce the performance index

$$J_r = \int_0^\tau (\gamma^{-1} z^T z - \gamma\omega^T\omega)dt,$$

since $x(0) = 0$, for $\forall \omega \in L_2[0,\infty]$ we can get

$$J_\tau = \int_0^\tau (\gamma^{-1} z^T z - \gamma\omega^T\omega + \dot{V}(t))dt - V(x(\tau)) \leqslant$$
$$\int_0^\tau (\gamma^{-1} z^T z - \gamma\omega^T\omega + \dot{V}(t))dt,$$

Where

$$\dot{V}(x) = x^T(t)(PA_i + A_i^T P)x(t) + 2x^T(t)PA_{di}x(t-d) + 2x^T(t)PB_{1i}\omega(t) + x^T(t)Sx(t) - x^T(t-d)Sx(t-d) =$$
$$\begin{bmatrix} x(t) \\ \omega(t) \\ x(t-d) \end{bmatrix}^T \begin{bmatrix} A_i^T P + PA_i + S & PB_i & PA_{di} \\ B_i^T P & 0 & 0 \\ A_{di}^T P & 0 & -S \end{bmatrix} \begin{bmatrix} x(t) \\ \omega(t) \\ x(t-d) \end{bmatrix}$$

$$\gamma^2 z^T z - \gamma\omega^T\omega =$$
$$\begin{bmatrix} x(t) \\ \omega(t) \\ x(t-d) \end{bmatrix} \begin{bmatrix} \gamma^{-1} C_{1i}^T C_{1i} & \gamma^{-1} C_{1i}^T D_i & 0 \\ \gamma^{-1} D_1^T C_{1i} & \gamma^{-1} D_i^T D_i - \gamma I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \omega(t) \\ x(t-d) \end{bmatrix}.$$

By Schur Complement Formula, matrix inequality (5) is equivalent to the following:

$$\begin{bmatrix} A_i^TP + PA_i + S + \gamma^{-1}C_{1i}^TC_{1i} & PB_i + \gamma^{-1}C_{1i}^TD_i & PA_{di} \\ B_i^TP + \gamma^{-1}D_i^TC_{1i} & \gamma^{-1}D_i^TD_i - \gamma I & 0 \\ A_{di}^TP & 0 & -S \end{bmatrix} < 0.$$

Then

$$J_r \leqslant \int_0^\tau (\gamma^{-1}z^Tz - \gamma\omega^T\omega + \dot{V}(t)dt =$$

$$\int_0^\tau \begin{bmatrix} x(t) \\ \omega(t) \\ x(t-d) \end{bmatrix}^T \prod_i \begin{bmatrix} x(t) \\ \omega(t) \\ x(t-d) \end{bmatrix} \mathrm{d}t < 0,$$

where

$$\prod_i = \begin{bmatrix} A_i^TP + PA_i + S + \gamma^{-1}C_{1i}^TC_{1i} & PB_i + \gamma^{-1}C_{1i}^TD_i & PA_{di} \\ B_i^TP + \gamma^{-1}D_i^TC_{1i} & \gamma^{-1}D_i^TD_i - \gamma I & 0 \\ A_{di}^TP & 0 & -S \end{bmatrix},$$

It can be shown that for $\forall\tau > 0$,

$$\int_0^\tau z^Tzdt < \gamma^2\int_0^\tau \omega^T\omega dt \leqslant \gamma^2\int_0^\infty \omega^T\omega dt,$$

Hence, $\| z(t) \|_2 < \gamma \| \omega(t) \|_2$ for $\forall \omega \in L_2[0,\infty]$. This concludes the proof.

Theorem 2: Given a constant $\gamma > 0$, the switched state feedback H_∞ control of system (2) is feasible if there exist symmetric matrices $\boldsymbol{X} > 0$, $\boldsymbol{Q} > 0$ and $\boldsymbol{Y}_i$, $(i \in M)$ such that the following linear matrix inequalities

$$\begin{bmatrix} (A_iX + B_{2i}Y_i)^T + A_iX + B_{2i}Y_i + Q & B_{1i} & XC_{1i}^T & A_{di}X \\ B_{1i}^T & -\gamma I & D_i^T & 0 \\ C_{1i}X & D_i & -\gamma I & 0 \\ XA_{di}^T & 0 & 0 & Q \end{bmatrix} < 0 \tag{6}$$

are satisfied. The state feedback gain matrices are given by

$$\boldsymbol{K}_i = \boldsymbol{Y}_i\boldsymbol{X}^{-1}.$$

Proof: By theorem 1, the switched state feedback H_∞ control of system (2) is feasible if there exist matrices $\boldsymbol{P}$ and $\boldsymbol{S}$ such that

$$\begin{bmatrix} A_i + B_{2i}K_i)^TP + P(A_i + B_{2i}K_i) + Q & PB_{1i} & C_{1i}^T & PA_{di} \\ B_{1i}^TP & -\gamma I & D_i^T & 0 \\ C_{1i} & D_i & -\gamma I & 0 \\ A_{di}^TP & 0 & 0 & S \end{bmatrix} < 0, \tag{7}$$

Multiplying the above inequality on both sides by $\mathrm{diag}\{P^{-1}, I, I, P^{-1}\}$. Denote $\boldsymbol{X} = \boldsymbol{P}^{-1}$, $\boldsymbol{Y}_i = \boldsymbol{K}_i\boldsymbol{P}^{-1}$ and $\boldsymbol{Q} = \boldsymbol{P}^{-1}\boldsymbol{SP}^{-1}$, then matrix inequality (7) is equivalent to matrix inequality (6). This concludes the proof.

Theorem 3: Given a constant $\gamma > 0$, the switched output feedback H_∞ control of system (3) is feasible if there exist symmetric matrices $\boldsymbol{X} > 0$, $\boldsymbol{Q} > 0$ and $\boldsymbol{U}_i$, $\boldsymbol{V}_i$, $(i \in M)$ such that the following linear matrix inequalities

$$\begin{bmatrix} (A_iX + B_{2i}U_iC_{2i})^T + A_iX + B_{2i}U_iC_{2i} + Q & B_{1i} & XC_{1i}^T & A_{di}X \\ B^{T1i} & -\gamma I & D_i^T & 0 \\ C_{1i}X & D_i & -\gamma I & 0 \\ XA_{di}^T & 0 & 0 & Q \end{bmatrix} < 0 \tag{8}$$

are satisfied. The output feedback gain matrices are given by $\boldsymbol{F}_i = \boldsymbol{U}_i\boldsymbol{V}_i^{-1}$ and $\boldsymbol{V}_i\boldsymbol{C}_{2i} = \boldsymbol{C}_{2i}\boldsymbol{X}$.

Proof: By theorem 1, the switched output feedback H_∞ control of system (3) is feasible if there exist matrices $\boldsymbol{P}$ and $\boldsymbol{S}$ such that

$$\begin{bmatrix} (A_i + B_{2i}K_iC_{2i})^TP + P(A_i + B_{2i}F_iC_{2i}) + Q & PB_{1i} & C_{1i}^T & PA_{di} \\ B_{1i}^TP & -\gamma I & D_i^T & 0 \\ C_{1i} & D_i & -\gamma I & 0 \\ A_{di}^TP & 0 & 0 & S \end{bmatrix} < 0 \tag{9}$$

Pre- and post-multiply the left-hand side (9) by diag $\{P^{-1}, I, I, P^{-1}\}$. Note that $\boldsymbol{X} = \boldsymbol{P}^{-1}$ and $\boldsymbol{Q} = \boldsymbol{P}^{-1}\boldsymbol{SP}^{-1}$, then we get

$$\begin{bmatrix} (A_iX + B_{2i}K_iC_{2i}X)^T + A_iX + B_{2i}F_iC_{2i}X + Q & B_{1i} & XC_{1i}^T & A_{di}X \\ B_{1i}^T & -\gamma I & D_i^T & 0 \\ C_{1i}X & D_i & -\gamma I & 0 \\ XA_{di}^T & 0 & 0 & Q \end{bmatrix} < 0. \tag{10}$$

Because C_{2i} is full row rank and X is positive-definite, it follows from $V_iC_{2i} = C_{2i}X$ that V_i is also full rank, and thus invertible. Because $V_iC_{2i} = C_{2i}X$ and $F_i = U_iV_i^{-1}$, we have $U_iC_{2i} = F_iC_{2i}X$. Replacing U_iC_{2i} in (8) by $F_iC_{2i}X$ gains matrix inequalities (10). This concludes the proof.

4 Numerical example

Consider the following switched systems composed of two subsystems:

$$\dot{x}(t) = \boldsymbol{A}_ix(t) + \boldsymbol{A}_{di}x(t-d) + \boldsymbol{B}_{1i}\omega(t) + \boldsymbol{B}_{2i}u(t),$$

$$z(t) = \boldsymbol{C}_{1i}x(t) + \boldsymbol{D}_i\omega(t),$$

$$y(t) = C_{2i}x(t) \quad (i = 1,2).$$

where

$$\boldsymbol{A}_1 = \begin{bmatrix} -2.5 & 1 \\ 0 & -3.6 \end{bmatrix}, \boldsymbol{A}_{d1} = \begin{bmatrix} 0.2 & 0.6 \\ 0.6 & 0.5 \end{bmatrix},$$

$$\boldsymbol{B}_{11} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}, \boldsymbol{B}_{21} = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.1 \end{bmatrix},$$

$$\boldsymbol{C}_{21} = \begin{bmatrix} 0.9 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}, \boldsymbol{C}_{11} = [1 \quad 0],$$

$$\boldsymbol{D}_1 = 0.5,$$

$$\boldsymbol{A}_2 = \begin{bmatrix} -1.7 & 1 \\ 0 & -2 \end{bmatrix}, \boldsymbol{A}_{d2} = \begin{bmatrix} 1 & 1 \\ 0.5 & 1.2 \end{bmatrix},$$

$$\boldsymbol{B}_{12} = \begin{bmatrix} 0.7 \\ 0.2 \end{bmatrix}, \boldsymbol{B}_{22} = \begin{bmatrix} 0.5 & 0.4 \\ 0.8 & 0.3 \end{bmatrix},$$

$$\boldsymbol{C}_{22} = \begin{bmatrix} 0.4 & 0.9 \\ 0.6 & 0.7 \end{bmatrix}, \boldsymbol{C}_{12} = [0 \quad 1],$$

$$\boldsymbol{D}_2 = 0.25.$$

Time delay $d = 0.5$. Given $\gamma = 0.5$, by using theorem 3 and Matlab-LMI toolbox, we can obtain the following symmetric positive-define matrices

$$\boldsymbol{X} = \begin{bmatrix} 0.0688 & -0.0763 \\ -0.0763 & 0.2882 \end{bmatrix}, \boldsymbol{Q} = \begin{bmatrix} 0.1927 & -0.0057 \\ -0.0057 & 0.2157 \end{bmatrix},$$

and matrices

$$\boldsymbol{U}_1 = \begin{bmatrix} 0.0057 & 0.0285 \\ 0.0156 & 0.0347 \end{bmatrix}, \boldsymbol{U}_2 = \begin{bmatrix} 0.0063 & 0.0085 \\ 0.0077 & 0.0086 \end{bmatrix},$$

$$\boldsymbol{V}_1 = \begin{bmatrix} 0.0247 & 0.0459 \\ -0.1264 & 0.0023 \end{bmatrix}, \boldsymbol{V}_2 = \begin{bmatrix} 0.6389 & -0.4945 \\ 0.3926 & -0.2819 \end{bmatrix}.$$

Then, the output feedback gain matrices are

$$\boldsymbol{F}_1 = \begin{bmatrix} 0.3921 & 0.0316 \\ 0.6828 & 0.0100 \end{bmatrix}, \boldsymbol{F}_2 = \begin{bmatrix} -0.3648 & 0.6098 \\ -0.3958 & 0.6637 \end{bmatrix}.$$

Therefore, the switched system with time delay is asymptotically stable with the expected disturbance attenuation performance for arbitrary switching.

5 Conclusion

In this paper, the problem of H_∞ stability analysis and control synthesis of time-delay switched systems has been addressed. The existence of a Lyapunov function to ensure H_∞ stability is proved to be equivalent to LMI-based conditions. Both the cases of state feedback and output feedback have been considered. All our results in this paper are expressed in terms of linear matrix inequalities, which can be easily solved by Matlab Toolbox.

References

[1] D. Liberzon, A. S. Morse, 1999. Basic problems in stability and design of switched systems. *IEEE Control Systems Magazine*, 19(5): 59-70.

[2] R. A. Decarlo, M. S. Branicky, Pettersson, et al, 2000. Perspectives and Results on the Stability and Stabilizability of Hybrid Systems. Proceedings of the IEEE, p. 1069-1082.

[3] R. W. Brockett, 1993. Hybrid Modes for Motion Control Systems. Essays in Control, Birkhauser, Boston.

[4] A. Gollu, P. P. Varaiya, 1989. Hybrid Dynamical Systems. Proceedings of the 28th IEEE Conference on Decision and Control, Tampa, p. 3228-3234.

[5] A. Back, J. Guckenheimer, M. A. Myers, 1993. Dynamical Simulation Facility for Hybrid Systems. Hybrid Systems, New York, Springer.

[6] G. Xie, L. Wang, Q. Ye, 2003. Controllability of a class of hybrid dynamic systems (Ⅰ)-Basic properties and preliminary results. *Applied Mathematics and Mechanics (English Edition)*, 24(9): 1041-1050.

[7] G. Xie, L. Wang, Q. Ye, 2003. Controllability of a class of hybrid dynamic systems (Ⅱ)-Single time-delay case. *Applied Mathematics and Mechanics (English Edition)*, 24(9): 1051-1062.

[8] G. Xie, L. Wang, Q. Ye, 2003. Controllability of a class of hybrid dynamic systems(Ⅲ)-Multiple time-delay case. *Applied Mathematics and Mechanics (English Edition)*, 24(9): 1063-1074.

[9] G. Zhai, Y. Sun, X. Chen, A. N. Michel, 2003. Stability and 2 L gain analysis for switched symmetric systems with time delay. Proceedings of the 2003 American Control conference, Denver, June, p. 4-6.

[10] J. P. Hespaha, 2003. Root-mean-square gains of switched linear systems. *IEEE Transaction on Automatic Control*, 348 (11): 2040- 2045.

[11] G. Zhai, B. Hu, K. Yasuda, et al., 2001. Disturbance attenuation properties of time-controlled switched systems. *J. Franklin Inst.*, 338: 765-779.

[12] H. Nie, J. Zhao, 2003. Hybrid state feedback H_∞ robust control for a class of linear systems with time-varying norm bounded uncertainty. *Proceedings of the American Control Conference*, Denver, 3608-3613.

[13] S. Hara, B. D. Anderson, H. Fujioka, 1994. Relating H_2 and H_∞ Norm Bounds for hybrid systems. Proceedings of the 33rd IEEE Conference on Decision and Control, Lake Buena Vista, FL, p. 7234-729.

[14] S. G. Lall, G. E. Dullerud, 1997. H_∞ synthesis for multi-rate hybrid systems. Proceedings of the 36th IEEE Conference on Decision and Control, San Diego, CA, p. 7234-7239.

[15] Z. J. Ji, X. X. Guo, L. Wang, et al, 2006. Robust H_∞ control and stabilization of uncertain switched linear systems: a multiple Lyapunov functions approach. *Transations of the ASME*, 128(3).

[16] Y. Song, J. Fan, M. R. Fei, et al, 2008. Robust H_∞ control of discrete switched system with time delay. *Applied Mathematics and Computation*, 205: 159-169.