Detecting Unstable Periodic Orbits in Hyperchaotic Systems Using Subspace Fixed-Point Iteration
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Abstract — We present a numerical method for efficiently detecting unstable periodic orbits (UPO's) embedded in chaotic attractors of high-dimensional systems. This method, which we refer to as subspace fixed-point iteration, locates fixed points of Poincaré maps using a form of fixed-point iteration that splits the phase space into appropriate subspaces. In this paper, among a number of possible implementations, we primarily focus on a subspace method based on the Schmelcher-Diakonos (SD) method that selectively locates UPO's by varying a stabilizing matrix, and present some applications of the resulting subspace SD method to hyperchaotic attractors where the UPO's have more than one unstable direction.

Keywords — nonlinear dynamics; chaos; unstable periodic orbits; numerical analysis; subspaces

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1 Introduction

The detection and quantification of embedded unstable periodic orbits (UPO's) is a crucial step in the analysis of chaotic attractors. While most analyses based on the periodic orbit theory have so far been performed for low-dimensional systems, some of recent studies have focused on higher-dimensional systems such as fluid dynamical systems [1]. In the future, improved understandings of UPO's in such high-dimensional systems will become practically significant for designing high-dimensional systems and developing high-dimensional control algorithms.

Thus the importance arises of developing efficient numerical methods for detecting UPO's in chaotic systems, and various attempts have been made including discussions on suitable damping coefficients for Newton-based methods [2]. One of popularly adopted proposals is a general method based on fixed-point iteration by Schmelcher and Diakonos (SD) [3], and this method relaxes the difficulty of Newton's method that a very good initial guess is necessary for the iterative procedure. The SD method uses the simple iteration

\[ x_{k+1} = x_k + \lambda C [F(x_k) - x_k] \] (1)

for locating fixed points of the (Poincaré) map \( F(x) \). Here, \( \lambda \) is a small positive number, and the matrix \( C \) contains only one nonzero entry (+1 or -1) per row or column. Since the SD method selectively locates (stabilizes) UPO's by varying \( C \), larger convergence basins for individual UPO's can be obtained than damped Newton's method. An extension of the SD method has been proposed by Davidchack and Lai (DL) [4], and the DL method is essentially a gradual transition from the SD method to Newton's method.

Applying either of the SD and DL methods to high-dimensional systems requires careful selection of the matrix \( C \). Moreover, applying the DL method to those systems requires a large amount of computation for constructing the Jacobian matrix. However, if the dynamical behavior of the target UPO is essentially low-dimensional, the difficulties can be greatly relaxed by introducing the approach of using invariant subspaces corresponding to repelling or weakly attracting directions around the orbit [5-8].

Thus we first proposed in [9] subspace projection methods (subspace fixed-point iteration) for easing the selection of the matrix \( C \) in applying the SD and DL methods to high-dimensional systems, and also for reducing the amount of computation required for integrating variational equations in the DL method. Since then the application and examination of this method have been extended to various systems and UPO's [10]. More recently, we have been discussing the fine structure of the convergence basins and its possible application to the sequential detection of UPO's [11].

In most of these studies, however, we have dealt with UPO's with a single unstable direction. Thus, as a further step, this paper focuses on some aspects of global and local convergence properties of the subspace SD and damped subspace Newton methods when applied to hyperchaotic attractors where the UPO's have more than one unstable direction.

2 Subspace fixed-point iteration
Among a number of possible implementations of subspace fixed-point iteration, we here focus on the subspace SD method and the closely related, damped subspace Newton method.

2.1 Subspace SD method

This method projects the SD method onto the low-dimensional invariant subspace and also exploits the stability in the orthogonal complement of the invariant subspace. In the subspace SD method, we first set a dimension $d$ so that all directions (around the target UPO's) that cannot be stabilized with $C=I$ are contained in the $d$-dimensional invariant subspace corresponding to the first $d$ (from the largest in magnitude) eigenvalues of the Jacobian matrix $DF(x)$. We then perform the iteration

$$x_{k+1} = x_k + \lambda \sum_{j=1}^{d} a_j \xi_j + \lambda \left[ F(x_k) - x_k - \sum_{j=1}^{d} b_j \xi_j \right]$$

(2)

where

$$'a_1, \ldots, a_d' = \hat{C}'(b_1, \ldots, b_d)$$

(3)

and

$$b_j = < F(x_k) - x_k, \xi_j > \quad \text{(inner product).} \quad (4)$$

Here, $\xi_i$'s represent basis vectors that span the $d$-dimensional invariant subspace, and they are obtained via subspace iteration or simultaneous iteration which is a multidimensional extension of the power method for computing the dominant eigenvalue and the corresponding eigenvector of a matrix. Further details of subspace iteration can be found in [6].

2.2 Damped subspace Newton method

When $\hat{C}$ is replaced with $-(\Phi - I)^{-1}$ where $\Phi$ is the $d \times d$ matrix whose elements $\phi_{ij}$ satisfy

$$DF(x_k) \xi_j = \sum_{j=1}^{d} \phi_{ij} \xi_j,$$

(5)

the subspace SD method becomes a damped version of subspace shooting [5,6] that efficiently locates UPO's of high-dimensional systems with essentially low-dimensional dynamics. In this paper, we refer to this method as damped subspace Newton method.

In the following numerical examples for both of the subspace SD and damped subspace Newton methods, we have introduced a modified iteration

$$x_{k+1} = x_k + (x_k - x_{k-1})$$

(6)

for the points $x_k$ where the linearized mapping $\xi \leadsto DF(x_k) \xi$ does not have a $d$-dimensional invariant subspace.

3 Convergence basins

To illustrate the basic properties of convergence basins of the subspace SD method, we here apply the method with $d=1$, $\hat{C} = -1$, and $\lambda = 0.002$ to the three-times iterated Ikeda map [12]

$$\begin{cases}
    x_{n+1} = 1 + 0.9 (x_n \cos w_n - y_n \sin w_n) \\
    y_{n+1} = 0.9 (x_n \sin w_n + y_n \cos w_n)
\end{cases}$$

(7)

where

$$w_n = 0.4 - 6/(1 + x_n^2 + y_n^2)$$

(8)

that has seven fixed points within its chaotic attractor. Fig. 1 shows the convergence basins for the three fixed points that can be detected by $\hat{C} = -1$. For comparison, we also present in Fig. 2 the convergence basins of the damped subspace Newton method with $d=1$ and $\lambda = 0.002$ for the same three fixed points. It is clearly seen from these figures that the enlargement of convergence basins by the SD method is well retained in the subspace SD method.

4 Application to hyperchaotic systems

Hyperchaos refers to the chaotic phenomenon that is associated with more than one positive Lyapunov exponent. For continuous-time autonomous systems, the minimal dimension of the phase space that allows hyperchaos is four, and in what follows we use the four-dimensional system [13]

$$\begin{cases}
    \dot{x} = -y - z \\
    \dot{y} = x + 0.25 y + w \\
    \dot{z} = 3.0 + x z \\
    \dot{w} = -0.5 z + 0.05 w
\end{cases}$$

(9)

as a test bed for the application of the subspace methods with $d=2$. Fig. 3 shows the hyperchaotic attractor of the four-dimensional system projected onto the $xzw$ space. The Poincaré section is defined as the three-dimensional hyperplane $y = 0$, and the Poincaré mapping point is defined as the point where the orbit crosses the Poincaré section from the $y > 0$ half space to the $y < 0$ half space. The resulting Poincaré mapping points are shown in Fig. 4. In what follows, we will detect the fixed points of the three-times iterated Poincaré map.

First we apply the damped subspace Newton method with $d=2$ and $\lambda = 0.1$. Figs. 5 and 6 show
a convergence process projected onto the xzw space and the zw plane, respectively. In these figures, the iteration starts from the initial, red point \((x, y, z, w) = (-30, 0, 1, 20)\), then proceeds with the movements (corrections) along the \(d\)-dimensional invariant subspace (blue segments) and its orthogonal complement (pink segments), and in the end converges to the green, fixed point \((x, y, z, w) = (-29.84, 0, 0.10, 15.04)\). In this example, we consider the convergence process to reflect a rather complex structure of the attractor.

We next investigate the convergence basins of the subspace methods applied to the hyperchaotic attractor. Here we reduce \(\lambda\) to 0.01, take 1500 points on the attractor as the initial points, and observe which initial point converges to which fixed point. Among the obtained results, Figs. 7 and 8 show the convergence basins for the damped subspace Newton method and the subspace SD method, respectively.

Here, the matrix \(\hat{C}\) for the subspace SD method is set to

\[
\hat{C} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\]

that detects fixed points (circles in the figures) with a dominant eigenvalue greater than unity and a second dominant eigenvalue also greater than unity. Then, for comparison, the convergence basins for both methods are plotted only for such fixed points. From the figures and the counted numbers of initial points that converge to the relevant fixed points, the enlargement of convergence basins, in the sense of average, by the subspace SD method is seen to be retained for the hyperchaotic attractor.

### 5 Concluding remarks

For much higher-dimensional systems that are the main target of the subspace methods, there is virtually
no choice of carpet-bombing the fixed points from a hyper-rectangular subregion of the phase space (as in Section 3), and therefore it is important to evaluate the convergence basins restricted on the attractor as we have seen in Section 4. However, we would like to mention that, from the viewpoint of understanding the underlying basin structure, further analysis by the carpet-bombing from a 3-dimensional rectangular subregion for the present test-bed system would be informative, and our observation will be presented elsewhere.

Fig. 5 A convergence process of the damped subspace Newton method from the red, initial point to the green, fixed point (projection onto the xzw space).

Fig. 6 The same as Fig. 5 (projection onto the zw plane).

Fig. 7 Convergence basins of the damped subspace Newton method for the hyperchaotic attractor.

Fig. 8 Convergence basins of the subspace SD method for the hyperchaotic attractor.

References