Stabilizability, Observability and Detectability for Discrete Stochastic Systems

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Abstract – **This paper mainly discusses stabilizatbility, exact observability and exact detectability of discrete stochastic systems with both static and control dependent noise via the spectrum technique. The authors put forward a definition of the spectrum and give some theorems based on the spectrum. Then the relation between discrete generalized Lyapunov equation and discrete generalized algebraic Riccati equation is also analyzed.**

Key words – *spectrum technique*; *discrete stochastic systems*; *detectability*; *observability*

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1 Introduction

It is well-known stabilizability is an important concept in mordern control theory. Meanwhile, observability and detectability are also essential concepts in linear control theory. In deterministic systems, observability and the detectability of a system are equivalent to controllability and stabilizability of its dual system respectively. In recent years, some fundamental concepts of deterministic system theory have been extended to stochastic $\hat{Ito}$ systems by many researchers[1-3].

For example, stochastic stabilizability is an essential assumption in many problems, such as infinite horizon stochastic optimal control problem[1], and the definitions of stochastic observability and detectability have been investigated extensively by many researchers, see Ref.[4-11]. Especially[9], extended complete observability to define "exact observability" of stochastic $\hat{Ito}$ system. In Ref.[6], the notion of observability leads to the stochastic version of the well-known rank criterion for observability of deterministic linear systems. Moreover, in order to generalize the results of stochastic H_2/H_∞ control theory to more general model with control or external disturbance dependent noise, the concept of exact detectability was introduced in Ref.[8].

As mentioned to the spectrum technique, it has been researched a lot in deterministic time-invariant system and the thoery is much more mature. In recent decades, some researchers have extended the spectrum technique to stochastic $\hat{Ito}$ system and different spectrum criterias for mean square stabilization and stability are also given[5,7,10,12]. In Ref.[7], the authors used the spectrum technique to study the stabilization and detectability of stochastic $\hat{Ito}$ system. Ref.[10] studied the stabilizability and exact observability of stochastic $\hat{Ito}$ systems with the aid of spectrum. But there are seldom studies for discrete stochastic systems via the spectrum technique.

In this paper, we mainly study the stabilizability, exact observability and exact detectability of the discrete stochastic system via the spectrum technique. Without loss of generality, we consider the following stochastic system with single nosie input:

$$\begin{cases} x(k+1) = [Ax(k) + Bu(k)] + \\ \qquad\qquad [Cx(k) + Du(k)]w(k), \\ x(0) = x_0 \in R^n,\ k = 1,2,3,\cdots,n. \end{cases} \tag{1}$$

where $x(k) \in R^n$, $u(k) \in R^m$ are the system state and the control, respectively. A, B, C, D are constant matrices of appropriate dimensions. $w(k) \in R$ is a sequences of real random variables defined on a complete probability space (Ω, F, μ), which is a wide sense stationary, second-order processes with $E(w(k)) = 0$ and $E(w(i)w(j)) = \delta_{ij}$. And δ_{ij} refers to the Kronecker function, i.e. $\delta_{ij} = 1$, if $i = j$ and $\delta_{ij} = 0$, if $i \neq j$.

On the other hand, we discuss the relation between generalized Lyapunov Equation (GLE) and Generalized Algebraic Riccati Equation (GARE). It is generally known that Lyapunov equation is useful in describing stability and stabilizability of deterministic systems. The classical results of traditional Lyapunov equation are extended to GLE by some researchers in the stochastic systems[7,8,13]. The Generalized Algebraic Riccati Equation (GARE) plays a pivotal role in the problem of indefinite stochastic linear quadratic optimal control and has been studied by some researchers[14-16].

For convenience, we adopt the following notations: we use s^n to denote the set of all $n \times n$ symmetric matrices; its components may be complex. A', A^* represent the transpose and complex conjugate transpose, respectively. $\boldsymbol{P} \geqslant (> 0)$ means $\boldsymbol{P}$ is a semi-positive (positive) definite matrix. We use $\sigma(L)$ to denote its spectral set of

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the operator or matrix $\boldsymbol{L}$. The symbol $\overline{p,q}$ where p and q are two integers denotes the set $\{p, p+1, \cdots, q\}$. Finally, we define a circular region: $C^{\odot} = \{s : |s| < 1\}$.

2 Spectrum of discrete stochastic systems

In this section, we give the definition of the spectrum and obtain the criterion for unremovable spectrum.

Considering system (1), for any $k \in \overline{1,N}$, taking $X(k) = E[x(k)x(k)']$ with state feedback control $u(k) = Kx(k)$, we obtain the following equation:

$$X(k+1) = (A+BK)X(k)(A+BK)' + (C+DK)X(k)(C+DK)'. \quad (2)$$

Inspired by Eq. (2) we give a definition of a linear operator and its spectrum.

Definition 1: For any feedback gain matrix $\boldsymbol{K}$, we define a linear operator Γ_k associated with the discrete stochastic system (1) as follows:

$$\Gamma_k = (A+BK)Z(A+BK)' + (C+DK)Z(C+DK)', \quad (3)$$

where $Z \in S^n$ and the spectral set of Γ_k is the set defined by

$$\sigma(\Gamma_k) = \{\lambda \mid \Gamma_k(Z) = \lambda Z, Z \neq 0, Z \in S^n\}. \quad (4)$$

Remark 1: It is easily seen that the following operator $\Gamma_k^* = (A+BK)'Z(A+BK) + (C+DK)'Z(C+DK)Z \in S^n$, is the adjoint operator of Γ_k with the inner product $< X, Y > = \mathrm{trace}(X^* Y)$ for any $X, Y \in S^n$. As we limit the coefficients to real matrices, so $\sigma(\Gamma_k^*) = \sigma(\Gamma_k)$.

Associated with the definition of spectrum, we give another definition of unremovable spectrum, which is an extended form of the stochastic *Itô* differential system[10].

Definition 2: λ is an unremovable spectrum of discrete stochastic system (1) with state feedback control $u(k) = Kx(k)$, if there exists $Z \neq 0 \in S^n$, such that for any $K \in R^{m\times n}$.

$$(A+BK)'Z(A+BK) + (C+DK)'Z(C+DK) = \lambda Z. \quad (5)$$

Below we give a criterion for the unremovable spectrum.

Theorem 1: λ is an unremovable spectrum of system (1) if there exists $Z \neq 0 \in S^n$, such that the following three equalities

$$A'ZA + C'ZC = \lambda Z, \; A'ZB + C'ZD = 0,$$
$$B'ZB + D'ZD = 0.$$

hold.

Proof: It is easy to see that Eq. (5) can be written as

$$A'ZA + C'ZC + (A'ZB + C'ZD)K + K'(A'ZB + C'ZD)' + K'(B'ZB + D'ZD)K = \lambda Z, \quad (6)$$

and the sufficiency is easily proven. In order to complete the proof, let $K = 0$ in Eq. (5), then

$$A'ZA + 'CZC = \lambda Z. \quad (7)$$

holds. From the proof of sufficient part, it follows

$$(A'ZB + C'ZD)K + K'(A'ZB + C'ZD)' + K'(B'ZB + D'ZD)K = 0. \quad (8)$$

Let $A'ZB + C'ZD = F$, $B'ZB + D'ZD = G$, then Eq. (8) becomes

$$FK + K'F' = -K'GK. \quad (9)$$

Since the left hand of Eq. (9) is linear with $\boldsymbol{K}$, we must have $G = 0$. Because of the arbitrarity of $\boldsymbol{K}$, we can obtain $\boldsymbol{F} = 0$ immediately by taking $K = F'$. So we complete the proof of this theorem.

3 Stabilizability of discrete stochastic systems

Below, we deal with the stabilizability of discrete stochastic system via the spectrum technique. In order to obtain the results on stabilizability, we firstly cite the following definitions:

Definition 3[11]: The following stochastic discrete-time system

$$x(k+1) = Ax(k) + Cx(k)w(k) \quad (10)$$

is called mean square stable if for any $x_0 \in R^n$, the corresponding state satisfies $\lim_{k\to\infty} \| Ex(k) \|^2$. If system (10) is stable, we also say (A, C) is stable for short.

Definition 4[11]: Stochastic system (1) is called stabilizable (in the mean square sense) if there exists a feedback control $u(k) = Kx(k)$, such that for any $x_0 \in R^n$, system (1) is mean square stable, i.e. $\lim_{k\to\infty} \| Ex(k) \|^2 = 0$.

It is well-known that system $x(k+1) = Ax(k) + Bu(k)$ is stabilizable if $\sigma(A+HK) \in C^{\odot}$, where $u(k) = Kx(k)$. Now we give the following theorem for stochastic case via the spectrum technique.

Theorem 2: The discrete stochastic system (1) is stabilizable if $\sigma(\Gamma_k) \in C^{\cdot}$.

Proof: We have obtained that for any $k \in \overline{1,N}$, the following equation holds:

$$\begin{cases} X(k+1) = (A+BK)X(k)(A+BK)' + (C+DK)X(k)(C+DK)', \\ X(0) = x(0)x'(0). \end{cases} \quad (11)$$

Since $X(k) \in S^n$, for any $X = (Ex_i x_j)_{n\times n} = (X_{ij\,n\times} \in S^n$, if we denote

$$\widetilde{X} = (X_{11}, X_{12}, \cdots, X_{1n}, X_{22}, \cdots, x_{2n}, \cdots, X_{nn})',$$

then there exists a unique matrix $\boldsymbol{\Gamma}_k \in R^{\frac{n(n+1)}{2}\times\frac{n(n+1)}{2}}$, that we can rewrite equation (11) as

$$\widetilde{X}(k+1) = \widetilde{\Gamma}_k \widetilde{\boldsymbol{X}}(k), \; \widetilde{X}(0) = \widetilde{X}_0,$$

Obviously

$$\lim_{k\to\infty} \| Ex(k) \|^2 = 0 \Leftrightarrow \lim_{k\to\infty} \widetilde{X}(k) = 0 \Leftrightarrow \sigma(\widetilde{\Gamma}_k) \in C^{\odot}.$$

This is obtained mainly because of the basic theory of deterministic systems. By Definition 1, it is easily proven that $\sigma(\Gamma_k) = \sigma(\widetilde{\Gamma}_k)$, so the proof of Theorem 2 is completed.

Remark 2: By Remark 1 and Theorem 2, it is easily seen system (1) is stabilizable iff there exists a $K \in R^{m\times n}$, such that $\sigma(\Gamma_k^*) \in C^{\odot}$.

In order to illustrate the spectrum and the proof of Theorem 2, we give an example as follows:

Example 1: In system (1), we let

$$\boldsymbol{A} = \begin{bmatrix} -0.5 & 0.1 \\ 0.2 & -0.5 \end{bmatrix},\ \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\boldsymbol{C} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix},\ \boldsymbol{D} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix},\ \boldsymbol{K} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then by a simple calculation, we obtain

$$\tilde{\boldsymbol{X}}(k+1) = \begin{bmatrix} X_{11}(k+1) \\ X_{12}(k+1) \\ X_{22}(k+1) \end{bmatrix} = \tilde{\boldsymbol{\Gamma}}_k \tilde{\boldsymbol{X}}(k) = \begin{bmatrix} 0.41 & 0.1 & 0.01 \\ 0.1 & 0.27 & 0.05 \\ 0.04 & 0.2 & 0.25 \end{bmatrix} \begin{bmatrix} X_{11}(k) \\ X_{12}(k) \\ X_{22}(k) \end{bmatrix}.$$

By a simple computation, we have

$$\sigma(\Gamma_k) = \sigma(\tilde{\Gamma}_k) = \{0.4817, 0.3008, 0.1474\}.$$

4 Exact observability and exact detecability

Next, we discuss exact observability and exact detectability of the discrete stochastic system (1). Before starting the discussion, we firstly give a definition of exact observability similarly to Definition 5 of Ref. [10].

Definition 5: Consider the following discrete stochastic system with measurement equation

$$\begin{cases} x(k+1) = Ax(k) + Cx(k)wk), \\ x(0) = x_0 \in R^n, \\ y(k) = Ux(k). \end{cases} \tag{12}$$

We call $x \in R^n$ an unobservable state, if for $\forall k \in \overline{1,N}$, the corresponding output response always equals zero. Stochastic system (12) is called exactly observable, if there is no unobservable state except zero initial state, i.e. if $y(k) = 0$, a/s. $\forall k \in \overline{1,N} \Rightarrow x_0 = 0$. For simplicity, when system (12) is exactly ocservable, we also say $[A, C | U]$ is exactly observable.

The following theorem give a criterion for exact observability of the discrete stochastic system (12).

Theorem 3: $[A, C \mid U]$ is exactly observable if there doesn't exist $Z \neq 0 \in S^n$, such that

$$A'ZA + C'AC = \lambda Z,\ UZ = 0. \tag{13}$$

Proof: For any $k \in \overline{1,N}$, we have

$$X(k+1) = AX(k)A' + CX(k)C',$$
$$X(0) = x_0 x_0'. \tag{14}$$

By Definition 5, we know that $[A, C \mid U]$ is exactly observable if for any arbitrary $X_0 = x_0 x_0' \neq 0$, there exists a $k \in \overline{1,N}$ such that

$$Y(k) = E[y(k)y(k)'] = UX(k)U' \neq 0. \tag{15}$$

From the proof of Theorem 2, Eq. (14) is equivalent to

$$\tilde{X}(k+1) = \tilde{\Gamma}(A, C)X(k),$$

where $\tilde{\Gamma}(A, C)$ is induced by matrices A and C. In addition, due to $X(k) \geqslant 0$ for all $k \in \overline{1,N}$, so Eq. (15) is equivalent to

$$Y_1(k) = U(k) \neq 0, \tag{16}$$

which is equivalent to

$$\vec{Y}_1(k) = \vec{\Gamma}_U \tilde{X}(k) \neq 0. \tag{17}$$

So Eq. (12) is exactly observable if the deterministic system

$$\begin{cases} \tilde{X}(k+1) = \tilde{\Gamma}(A, C)X(k), \\ \vec{Y}_1 k) = \vec{\Gamma}_U \tilde{X}(k). \end{cases} \tag{18}$$

is completely observable. By the deterministic theory, (18) is completely observable if there doesn't exists an eigenvector $\xi \neq 0$ with $n(n+1)/2$ dimensions, such that

$$\tilde{\Gamma}(A, C)\xi = \lambda\xi,\ \tilde{\boldsymbol{\Gamma}}_U \xi = 0. \tag{19}$$

Obviously, Eq. (19) is equivalent to that there doesn't exists $Z \neq 0 \in S^n$ satisfying Eq. (13); the proof is completed.

Then we give the following example to demonstrate the notion of $\vec{\Gamma}_U$.

Example 2: In Eq. (16), take

$$\boldsymbol{Y}_1(k) = \begin{bmatrix} y_{11}(k) & y_{12}(k) \\ y_{21}(k) & y_{22}(k) \end{bmatrix},\ \boldsymbol{U} = \begin{bmatrix} 0.5 & 0.2 \\ 0.7 & -0.3 \end{bmatrix},$$

$$\boldsymbol{X}(k) = \begin{bmatrix} x_{11}(k) & x_{12}(k) \\ x_{12}(k) & x_{22}(k) \end{bmatrix}. \tag{20}$$

From $\boldsymbol{Y}_1(k) = U\tilde{X}(k)$, we have

$$\begin{cases} y_{11}(k) = 0.5x_{11}(k) + 0.2x_{12}(k), \\ y_{12}(k) = 0.5x_{12}(k) + 0.2x_{22}(k), \\ y_{21}(k) = 0.7x_{11}(k) - 0.3x_{12}(k), \\ y_{22}(k) = 0.7x_{12}(k) - 0.3x_{22}(k). \end{cases} \tag{21}$$

Then Eq. (21) can be written in the matrix form as

$$\vec{\boldsymbol{Y}}(k) = \begin{bmatrix} 0.5 & 0.2 & 0 \\ 0 & 0.5 & 0.2 \\ 0.7 & -0.3 & 0 \\ 0 & 0.7 & -0.3 \end{bmatrix} \tilde{X}(k) = \vec{\Gamma}_U \tilde{X}(k),$$

so

$$\vec{\boldsymbol{\Gamma}}_U = \begin{bmatrix} 0.5 & 0.2 & 0 \\ 0 & 0.5 & 0.2 \\ 0.7 & -0.3 & 0 \\ 0 & 0.7 & -0.3 \end{bmatrix}.$$

Now, we give the following definition of exact detectability of the discrete stochastic system (12).

Definition 6: System (12) or $[A, C \mid U]$ is said to be exactly detectable, if $y(k) = 0$, a.s, for any $k \in \overline{1,N}$ implies $\lim_{k\to\infty} \| Ex(k) \|^2$.

Definition 6 means that when the measurement output is identically zero, the corresponding state is asymptotically mean square stable. Next, we give the following stochastic PBH criterion for exact detectability in discrete case.

Theorem 4: $[A, C \mid U]$ is exactly detectable if there does not exist $Z \neq 0 \in R^n$ such that

$$\begin{cases} AZA' + CZC' = \lambda Z,\ |\lambda| \geqslant 1, \\ UZ = 0. \end{cases} \tag{22}$$

Proof: The proof is quite similar to Theorem 3 and thus omitted.

5 Relation between GLE and GARE

In this section, we consider the relation between GLE of the closed-loop discrete stochastic systems and GARE resulting from the discrete stochastic LQ control.

The following GLE of closed-loop system (1) with state feedback control $u(k) = Kx(k)$ is considered:

$$P = (A + BK)'P(A + BK) + (C + DK)'P(C + DK) + Q, \quad (23)$$

where $Q \geqslant 0$.

On the other hand, for system (1), the cost function in discrete LQ control is determined by

$$J(x_0, u) := \sum_{k=0}^{\infty} E[x'(k)Qx(k) + u'(k)Ru(k)], \quad (24)$$

where $R > 0$.

The following discrete GARE is important in discrete LQ control:

$$\begin{cases} P = A'PA + C'PC + Q - (A'PB + C'PD) \cdot \\ (R + B'PB + D'PD)^{-1}(A'PB + C'PD)', \\ R + B'PB + D'PD > 0. \end{cases} \quad (25)$$

In Ref. [11], it has shown under the assumptions of stabilizability and exact observability, the optimal control and the optimal value are associated with Eq. (25).

Then we give a theorem on relation between the above GLE and GARE:

Theorem 5: Define F_P is the solution set of Eq. (23) and S_P is the solution set of Eq. (25), then $F_P \supseteq S_P$.

Proof: Let $P \in S_P$, then $\exists\, Q \geqslant 0$ and $R > 0$ such that Eq. (25) holds. Then we define the matrix K in Eq. (23) as

$$K = -(R + B'PB + D'PD)^{-1}(A'PB + C'PD)'. \quad (26)$$

After some manipulations, Eq. (25) can be written as

$$\begin{cases} P = (A + BK)'P(A + BK) + \\ (C + DK)'P(C + DK) + Q + K'RK, \\ R + B'PB + D'PD > 0. \end{cases} \quad (27)$$

Because $R > 0$, it is easily to see that $Q + F'RF \geqslant 0$, then Eq. (27) is a GLE. Therefore $P \in F_P \Rightarrow F_P \supseteq S_P$.

Theorem 5 means that the set of all matrices that satisfy GLE of the closed-loop discrete system contains the set of all matrices that satisy the discrete GARE constructed by the discrete stochastic LQ problem.

6 Conclusions

In this paper, we have studied the stabilizability, exact observability and exact detectability of discrete stochastic systems via the spectrum technique. The necessary and sufficient conditions for unremovable spectrum, the stabilizability, exact detectability, exact observability of discrete stochastic system are presented by means of the spectrum of operator Γ_k respectively. Moreover, we consider the relation on solution between GLE and GARE.

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